Problem Set 11 Solution Set

Anthony Varilly

Math 112, Spring 2002

1. *End of Chapter 8, Exercise 24.* Give an example to show that the following is not equivalent to the integrability of f. For any $\epsilon > 0$, there is a $\delta > 0$ such that if P is any partition into rectangles $S_1, \ldots, S_N$ with sides $< \delta$, there exist $x_1 \in S_1, \ldots, x_N \in S_N$ such that

$$\left| \sum_{i=1}^{n} f\left(x_i\right) v\left(S_i\right) - I \right| < \epsilon.$$

Solution. Assume that the condition is equivalent to the integrability of f. Consider the function $f : [0,1] \to \mathbb{R}$, where $f(x) = 0$ if x is irrational and $f(x) = 1$ if x is rational. Then for any partition P into rectangles $S_1, \ldots, S_N$ with sides $< \delta$, there exist irrational points $x_i \in S_i$ at which $f(x_i) = 0$. Thus

$$\sum_{i=1}^{n} f\left(x_i\right) v\left(S_i\right) = 0$$

Then

$$\left| \sum_{i=1}^{n} f\left(x_i\right) v\left(S_i\right) - I \right| = |I| < \epsilon,$$

for any $\epsilon > 0$, if and only if $I = 0$. This implies that f is integrable with integral zero. But as we showed in class, f is not Riemann integrable since $L(f,P) = 0$ whereas $U(f,P) = 1$. $\square$

2. *End of Chapter 8, Exercise 36.* Prove that $\lim_{n\to\infty} \left(n!\right)^{1/n}/n = e^{-1}$ by considering Riemann sums for $\int_0^1 \log x dx$ based on the partition $1/n < 2/n < \cdots < 1$.

Solution. We first rewrite

$$\begin{aligned}
\frac{(n!)^{1/n}}{n} &= \left(\frac{n!}{n^n}\right)^{1/n} = \left(\frac{n\left(n-1\right)\left(n-2\right)\cdots 1}{n \cdot n \cdot \cdots \cdot n}\right)^{1/n} \\
&= \left(1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \cdots \cdot \frac{1}{n}\right)^{1/n} \\
&= \exp\left(\frac{1}{n}\sum_{k=0}^{n-1} \log\left(1 - \frac{k}{n}\right)\right)
\end{aligned}$$

If we let $f(x) = \log(1-x)$ and let $P_n = 1/n < 2/n < \cdots < 1$, then we recognize $(1/n)\sum_{k=0}^{n-1}\log\left(1-\frac{k}{n}\right)$ as the lower Riemann sum $L(f, P_n)$ on $[0,1]$. Now let us evaluate $\int_0^1 \log(1-x)dx$:

$$\begin{aligned}\int_0^1 \log(1-x)dx &= \int_0^1 \log y\, dy = \lim_{\epsilon\to 0}\int_\epsilon^1 \log y\, dy \\ &= \lim_{\epsilon\to 0} -1-(\epsilon\log\epsilon-\epsilon)\end{aligned}$$

Using l'Hôpital's Rule, we find that

$$\lim_{\epsilon\to 0}\epsilon\log\epsilon = \lim_{\epsilon\to 0}\frac{\log\epsilon}{1/\epsilon} = \lim_{\epsilon\to 0}\frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon\to 0}(-\epsilon) = 0$$

Therefore $\int_0^1 \log(1-x)dx = -1$. Since $f(x) = \log(1-x)$ is integrable on $[0,1]$, $L(f, P_n) = \frac{1}{n}\sum_{k=0}^{n-1}\log\left(1-\frac{k}{n}\right)$ must approach -1 as $n \to \infty$. Hence we conclude that

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n} = e^{-1}$$

as desired. ☐

3. *End of Chapter 8, Exercise 38.* Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = p/q \end{cases}$$

where $p, q \geq 0$ with no common factor. Show that f is integrable, and compute $\int_0^1 f$.

Solution. We first show that f is discontinuous at every rational number in $[0,1]$. Let $x \in [0,1]\cap\mathbb{Q}$ and choose $\epsilon < 1/q$. There exists an irrational number y such that $|x-y| < \delta$, which implies that

$$|f(x)-f(y)| = |1/q - 0| = |1/q| < \epsilon$$

Thus we have produced an ϵ for which there is no δ such that $|f(x)-f(y)| < \epsilon$ when $|x-y| < \delta$.

We now show that f is continuous at every irrational number in $[0,1]$. Let c be an irrational number in $[0,1]$ and $\epsilon > 0$. Then there is a natural number n such that $1/n < \epsilon$. If we choose δ small enough that the interval $(c-\delta, c+\delta)$ contains no rational numbers with denominator less than n, then it follows that for x in this interval we have $|f(x)-f(c)| = |f(x)| \leq 1/n < \epsilon$.

Thus we have shown that f is discontinuous on $D(f) = [0,1]\cap\mathbb{Q} \subset \mathbb{Q}$. Since $\mathbb{Q}$ is countable, we know that $D(f)$ is also countable. Since the interval $[0,1]$ is bounded and has volume, and since f is bounded by $1/1 = 1$, we have satisfied the conditions of Corollary 8.3.3, from which it follows that f is integrable.

Finally we show that $\int_0^1 f = 0$. Take any partition $P_n = \{x_0, x_1, \dots x_n\}$ and consider $d_j = \inf \{f(x) : x \in [x_{j-1}, x_j]\}$. Since every subinterval $[x_{j-1}, x_j]$ must contain irrational numbers, it follows that $d_j = 0$. Then

$$\underline{\int_0^1} f = \sup \{L(f, P_n)\} = 0$$

and since f was shown to be integrable, the upper and lower integrals must agree. ☐

4. *Exercise 10.1.2.* Show that

$$\left| \int_a^b f(x) dx \right|^2 \leq (b-a) \int_a^b |f(x)|^2 \, dx,$$

and deduce that a square integrable function on $[a, b]$, continuous on (a, b), is also integrable. Is the converse true?

Solution. We apply the Cauchy-Schwarz inequality to functions $f, g : [a, b] \to \mathbb{C}$, and we take $g(x) = 1$ for all $x \in [a, b]$. Then

$$|\langle f, g \rangle| \leq \|f\| \, \|g\|$$

implies

$$\left(\int_a^b f(x)\overline{g(x)} dx \right)^2 \leq \left(\int_a^b |f(x)|^2 \, dx \right) \left(\int_a^b |g(x)|^2 \, dx \right)$$

which leads to

$$\begin{aligned} \left(\int_a^b f(x) dx \right)^2 &\leq \left(\int_a^b |f(x)|^2 \, dx \right) \left(\int_a^b dx \right) \\ &= (b-a) \int_a^b |f(x)|^2 \, dx \end{aligned}$$

Thus if f is square integrable on $[a, b]$, then $\int_a^b |f(x)|^2 \, dx$ converges and so $\int_a^b f(x) dx$ converges as well by the above inequality.

The converse, however, is not true. Consider the function $f(x) = 1/\sqrt{x}$. Then f is continuous on $(0, 1)$, and

$$\int_0^1 f(x) dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2.$$

But

$$\int_0^1 |f(x)|^2 \, dx = \int_0^1 \frac{dx}{x} = \lim_{\epsilon \to 0} (\log 1 - \log \epsilon) = \infty$$

Therefore f is not square integrable even though f is integrable. ☐

5. *Exercise 10.2.2.* Let $g_0, g_1, g_2, \dots$ be linearly independent vectors in an inner product space. Inductively define

$$h_0 = g_0, \ \varphi_0 = \frac{h_0}{\|h_0\|}, \dots, \ h_n = g_n - \sum_{k=0}^{n-1} \langle g_n, \varphi_k \rangle \varphi_k, \ \varphi_n = \frac{h_n}{\|h_n\|} \dots.$$

Show that $\varphi_0, \varphi_1, \varphi_2, \dots$ are orthonormal. Why must we assume that the g's are linearly independent?

Solution. Without loss of generality, let $0 \leq k < j$. Then

$$\begin{aligned}
\langle \varphi_j, \varphi_k \rangle &= \left\langle \frac{g_n - \sum_{n=0}^{j-1} \langle g_j, \varphi_n \rangle \varphi_n}{\|g_n - \sum_{n=0}^{j-1} \langle g_j, \varphi_n \rangle \varphi_n\|}, \varphi_k \right\rangle \\
&= \frac{\langle g_j, \varphi_k \rangle - \langle g_j, \varphi_k \rangle \langle \varphi_k, \varphi_k \rangle}{\|g_n - \sum_{n=0}^{j-1} \langle g_j, \varphi_n \rangle \varphi_n\|} \\
&= 0
\end{aligned}$$

since

$$\langle \varphi_k, \varphi_k \rangle = \left\langle \frac{h_k}{\|h_k\|}, \frac{h_k}{\|h_k\|} \right\rangle = \frac{\langle h_k, h_k \rangle}{\|h_k\|^2} = 1$$

Thus $\langle \varphi_j, \varphi_k \rangle = \delta_{jk}$, so $\varphi_0, \varphi_1, \varphi_2, \dots$ are orthonormal.

We assume that the g's are linearly independent so that g_j cannot be written as a linear combination of $\{g_1, \dots, g_{j-1}\}$. This implies that g_j cannot be written as a linear combination of $\{\varphi_1, \dots, \varphi_{j-1}\}$, since each φ_i is a linear combination of $\{g_1, \dots, g_{j-1}\}$. Hence

$$\|h_j\| = \|g_n - \sum_{n=0}^{j-1} \langle g_j, \varphi_n \rangle \varphi_n\| \neq 0$$

and so each φ_j is well-defined. □

6. *Exercise 10.2.3.*

   (a) Suppose $\varphi_0(x), \varphi_1(x), \dots$ are orthonormal functions on $[0, 2\pi]$. Show that the functions

   $$\psi_n(x) = \sqrt{\frac{2\pi}{\ell}} \varphi_n\left(\frac{2\pi x}{\ell}\right)$$

   are orthonormal on $[0, \ell]$.

   Solution. We have to show that $\langle \psi_n, \psi_m \rangle = \delta_{nm}$. From the definition of the ψ's, we have

   $$\begin{aligned}
   \langle \psi_n, \psi_m \rangle &= \frac{2\pi}{\ell} \int_0^\ell \varphi_n\left(\frac{2\pi x}{\ell}\right) \overline{\varphi_m\left(\frac{2\pi x}{\ell}\right)} dx \\
   &= \frac{2\pi}{\ell} \int_0^{2\pi} \varphi_n(u) \overline{\varphi_m(u)} du \cdot \frac{\ell}{2\pi} \\
   &= \int_0^{2\pi} \varphi_n(u) \overline{\varphi_m(u)} du = \delta_{nm}
   \end{aligned}$$

   since $\varphi_0(x), \varphi_1(x), \dots$ are orthonormal functions on $[0, 2\pi]$. □

   (b) Write the family obtained by modifying $1/\sqrt{2\pi}$, $(\sin nx)/\sqrt{\pi}$, $(\cos mx)/\sqrt{\pi}$, or, alternatively, $e^{inx}/\sqrt{2\pi}$ to $[0, \ell]$ as in (a).

Solution. The function $\dfrac{1}{\sqrt{2\pi}}$ becomes $\sqrt{\dfrac{2\pi}{\ell}}\dfrac{1}{\sqrt{2\pi}} = \dfrac{1}{\sqrt{\ell}}$. The functions $\dfrac{\sin nx}{\sqrt{\pi}}$ become $\sqrt{\dfrac{2\pi}{\ell}}\dfrac{1}{\sqrt{\pi}}\sin\dfrac{2\pi n}{\ell}x = \sqrt{\dfrac{2}{\ell}}\sin\dfrac{2\pi n}{\ell}x$. The functions $\dfrac{\cos mx}{\sqrt{\pi}}$ become $\sqrt{\dfrac{2\pi}{\ell}}\dfrac{1}{\sqrt{\pi}}\cos\dfrac{2\pi m}{\ell}x = \sqrt{\dfrac{2}{\ell}}\cos\dfrac{2\pi m}{\ell}x$. ☐

(c) Write the Fourier series of f for the families obtained in (b).

Solution.

$$\begin{aligned}
f &= \left\langle f, \frac{1}{\sqrt{\ell}}\right\rangle \frac{1}{\sqrt{\ell}} + \sum_{n=1}^{\infty}\left\langle f, \sqrt{\frac{2}{\ell}}\sin\frac{2\pi n}{\ell}x\right\rangle \sqrt{\frac{2}{\ell}}\sin\frac{2\pi n}{\ell}x \\
&\quad + \sum_{m=1}^{\infty}\left\langle f, \sqrt{\frac{2}{\ell}}\cos\frac{2\pi m}{\ell}x\right\rangle \sqrt{\frac{2}{\ell}}\cos\frac{2\pi m}{\ell}x \\
&= \left\langle f, \frac{1}{\sqrt{\ell}}\right\rangle \frac{1}{\sqrt{\ell}} + \frac{2}{\ell}\sum_{n=1}^{\infty}\left\langle f, \sin\frac{2\pi n}{\ell}x\right\rangle \sin\frac{2\pi n}{\ell}x \\
&\quad + \frac{2}{\ell}\sum_{m=1}^{\infty}\left\langle f, \cos\frac{2\pi m}{\ell}x\right\rangle \cos\frac{2\pi m}{\ell}
\end{aligned}$$

☐

(d) Show that if the φ_n in (a) are complete, so are the ψ_n.

Solution. We will use Parseval's Theorem to establish completeness. Consider the quantity $\langle f, \psi_n\rangle$.

$$\begin{aligned}
\langle f, \psi_n\rangle &= \int_0^{\ell} f(x)\overline{\psi_n(x)}dx = \sqrt{\frac{2\pi}{\ell}}\int_0^{\ell} f(x)\overline{\varphi_n\left(\frac{2\pi x}{\ell}\right)}dx \\
&= \sqrt{\frac{2\pi}{\ell}}\cdot\frac{\ell}{2\pi}\int_0^{\ell} f\left(\frac{\ell u}{2\pi}\right)\overline{\varphi_n\,(u)}du \\
&= \sqrt{\frac{\ell}{2\pi}}\int_0^{\ell} g(u)\overline{\varphi_n(u)}du = \sqrt{\frac{\ell}{2\pi}}\langle g, \varphi_n\rangle
\end{aligned}$$

If the φ_n in (a) are complete, then by Parseval's relation, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}|\langle g, \varphi_n\rangle|^2 &= \|g\|^2 = \langle g, g\rangle \\
&= \int_0^{2\pi}\left|f\left(\frac{\ell u}{2\pi}\right)\right|^2 du = \frac{2\pi}{\ell}\int_0^{\ell}|f\,(y)|^2\,dy \\
&= \frac{2\pi}{\ell}\langle f, f\rangle = \frac{2\pi}{\ell}\|f\|^2
\end{aligned}$$

Putting these ideas together yields

$$\sum_{n=0}^{\infty} |\langle f, \psi_n \rangle|^2 = \frac{\ell}{2\pi} \sum_{n=0}^{\infty} |\langle g, \varphi_n \rangle|^2 = \frac{\ell}{2\pi} \cdot \frac{2\pi}{\ell} \|f\|^2 = \|f\|^2$$

Thus the ψ_n are complete by Parseval's Theorem. □

7. *Exercise 10.2.4.* Assume for the moment that the functions $1/\sqrt{2\pi}$, $(\sin nx)/\sqrt{\pi}$, $(\cos mx)/\sqrt{\pi}$ are complete on the interval $[0, 2\pi]$.

   (a) Apply this to the function x to show that $x = \pi - 2\sum_{n=1}^{\infty} (\sin nx)/n$.

   Solution. By completeness, we can write

$$x = \left\langle x, \frac{1}{\sqrt{2\pi}} \right\rangle \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left\langle x, \frac{\sin nx}{\sqrt{\pi}} \right\rangle \frac{\sin nx}{\sqrt{\pi}} + \sum_{m=1}^{\infty} \left\langle x, \frac{\cos mx}{\sqrt{\pi}} \right\rangle \frac{\cos mx}{\sqrt{\pi}}$$

   We now calculate the Fourier coefficients:

$$\begin{aligned} \left\langle x, \frac{1}{\sqrt{2\pi}} \right\rangle &= \int_0^{2\pi} \frac{x}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \frac{4\pi^2}{2} \\ \left\langle x, \frac{\sin nx}{\sqrt{\pi}} \right\rangle &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} x \sin(nx) dx = -\frac{1}{\sqrt{\pi}} \frac{2\pi}{n} \\ \left\langle x, \frac{\cos mx}{\sqrt{\pi}} \right\rangle &= \frac{1}{\sqrt{\pi}} \int_0^{2\pi} x \cos(mx) dx = 0 \end{aligned}$$

   It follows that

$$\begin{aligned} x &= \frac{1}{\sqrt{2\pi}} \frac{4\pi^2}{2} \cdot \frac{1}{\sqrt{2\pi}} - \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi}} \frac{2\pi}{n} \frac{\sin nx}{\sqrt{\pi}} \\ &= \pi - 2 \sum_{n=1}^{\infty} \frac{(\sin nx)}{n} \end{aligned}$$

   □

   (b) Using the Fourier coefficients found in (a), apply Parseval's relation to show that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.

   Solution. From Parseval's relation, we have

$$\|x\|^2 = \left| \left\langle x, \frac{1}{\sqrt{2\pi}} \right\rangle \right|^2 + \sum_{n=1}^{\infty} \left| \left\langle x, \frac{\sin nx}{\sqrt{\pi}} \right\rangle \right|^2$$

   Since

$$\|x\|^2 = \int_0^{2\pi} x^2 dx = \frac{8\pi^3}{3}$$

we obtain

$$\frac{8\pi^3}{3} = \frac{4\pi^4}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi}\frac{4\pi^2}{n^2}$$

or

$$\frac{8\pi^3}{3} = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty} \frac{4\pi^2}{n^2}$$

which leads to

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\square$

(c) Use the same procedure on x^2 to get $\pi^4/90 = \sum_{n=1}^{\infty} 1/n^4$.

Solution. From Parseval's relation, we have

$$\|x^2\|^2 = \left|\left\langle x^2, \frac{1}{\sqrt{2\pi}}\right\rangle\right|^2 + \sum_{n=1}^{\infty}\left|\left\langle x^2, \frac{\sin nx}{\sqrt{\pi}}\right\rangle\right|^2 + \sum_{m=1}^{\infty}\left|\left\langle x^2, \frac{\cos mx}{\sqrt{\pi}}\right\rangle\right|^2$$

We now compute the Fourier coefficients:

$$\begin{aligned}
\left\langle x^2, \frac{1}{\sqrt{2\pi}}\right\rangle &= \int_0^{2\pi} \frac{x^2}{\sqrt{2\pi}}dx = \frac{1}{\sqrt{2\pi}}\frac{8\pi^3}{3} \\
\left\langle x^2, \frac{\sin nx}{\sqrt{\pi}}\right\rangle &= \frac{1}{\sqrt{\pi}}\int_0^{2\pi} x^2\sin(nx)dx \\
&= \frac{1}{\sqrt{\pi}}\left[\frac{2x}{n^2}\sin nx - \frac{n^2x^2-2}{n^3}\cos nx\right]_0^{2\pi} = -\frac{1}{\sqrt{\pi}}\frac{4\pi^2}{n} \\
\left\langle x^2, \frac{\cos mx}{\sqrt{\pi}}\right\rangle &= \frac{1}{\sqrt{\pi}}\int_0^{2\pi} x^2\cos(mx)dx \\
&= \frac{1}{\sqrt{\pi}}\left[\frac{2x}{m^2}\cos mx + \frac{m^2x^2-2}{m^3}\sin mx\right]_0^{2\pi} = \frac{1}{\sqrt{\pi}}\frac{4\pi}{m^2}
\end{aligned}$$

Since

$$\|x^2\|^2 = \int_0^{2\pi} x^4 dx = \frac{32\pi^5}{5}$$

we obtain

$$\begin{aligned}
\frac{32\pi^5}{5} &= \left(\frac{1}{\sqrt{2\pi}}\frac{8\pi^3}{3}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{\pi}}\frac{4\pi^2}{n}\right)^2 + \sum_{m=1}^{\infty}\left(\frac{1}{\sqrt{\pi}}\frac{4\pi}{m^2}\right)^2 \\
&= \frac{32\pi^5}{9} + 16\pi^3\sum_{n=1}^{\infty}\frac{1}{n^2} + 16\pi\sum_{m=1}^{\infty}\frac{1}{m^4} \\
&= \frac{32\pi^5}{9} + \frac{16\pi^5}{6} + 16\pi\sum_{m=1}^{\infty}\frac{1}{m^4}
\end{aligned}$$

from which it follows that

$$\frac{\pi^4}{90} = \sum \frac{1}{m^4}.$$

□

8. *Exercise 10.2.5.* Prove that

$$2\sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[(n+1/2)\,\theta\right]}{\sin\left(\theta/2\right)} - 1$$

by using $e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \dfrac{e^{i\theta}\left(1-e^{in\theta}\right)}{1-e^{i\theta}}$.

Solution. Since $\cos k\theta = \operatorname{Re}\left(e^{i\theta}\right)^k$, we can rewrite $\sum_{k=1}^{n} \cos k\theta$ as

$$\sum_{k=1}^{n} \cos k\theta = \operatorname{Re}\left[\sum_{k=1}^{n}\left(e^{i\theta}\right)^k\right] = \operatorname{Re}\left[\frac{e^{i\theta}\left(1-e^{in\theta}\right)}{1-e^{i\theta}}\right]$$

Multiply numerator and denominator by the complex conjugate of $1 - e^{i\theta}$ to get

$$\begin{aligned}
\sum_{k=1}^{n} \cos k\theta &= \operatorname{Re}\left[\frac{e^{i\theta}\left(1-e^{in\theta}\right)\left(1-e^{-i\theta}\right)}{\left(1-e^{i\theta}\right)\left(1-e^{-i\theta}\right)}\right] \\
&= \operatorname{Re}\left[\frac{e^{i\theta} - 1 + e^{in\theta} - e^{i(n+1)\theta}}{\left(1-e^{i\theta}\right)\left(1-e^{-i\theta}\right)}\right] \\
&= \operatorname{Re}\left[-\frac{1-e^{i\theta}}{\left(1-e^{i\theta}\right)\left(1-e^{-i\theta}\right)}\right] + \operatorname{Re}\left[\frac{e^{in\theta}\left(1-e^{i\theta}\right)}{\left(1-e^{i\theta}\right)\left(1-e^{-i\theta}\right)}\right] \\
&= -\frac{1-\cos\theta}{2\left(1-\cos\theta\right)} + \operatorname{Re}\left[\frac{e^{in\theta}\left(1-e^{i\theta}\right)}{\left(1-e^{i\theta}\right)\left(1-e^{-i\theta}\right)} \cdot \frac{e^{i\theta/2}}{e^{i\theta/2}}\right] \\
&= -\frac{1}{2} + \operatorname{Re}\left[\frac{e^{i(n+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}}\right] \\
&= -\frac{1}{2} + \operatorname{Re}\left[\frac{\cos\left(n+1/2\right)\theta + i\sin\left(n+1/2\right)\theta}{2i\sin\left(\theta/2\right)}\right] \\
&= -\frac{1}{2} + \operatorname{Re}\left[\frac{-i\cos\left(n+1/2\right)\theta + \sin\left(n+1/2\right)\theta}{2\sin\left(\theta/2\right)}\right] \\
&= -\frac{1}{2} + \frac{1}{2} \cdot \frac{\sin\left(n+1/2\right)\theta}{\sin\left(\theta/2\right)}
\end{aligned}$$

Therefore

$$2\sum_{k=1}^{n} \cos k\theta = \frac{\sin\left[(n+1/2)\,\theta\right]}{\sin\left(\theta/2\right)} - 1$$

as desired. □