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Problem Set 10 Solutions
April 22, 2002

1 3

Given $\epsilon > 0$, set $N = \frac{2(f(b)-f(a))(b-a)}{\epsilon}$ and consider the $P_N = \{x_i = a + i\frac{b-a}{N}\}$ for $0 \leq i \leq N$.

Then $U(f,P_N) - L(f,P_N) = \frac{b-a}{N}(\sum_{i=1}^{N} f(x_i) - \sum_{i=0}^{N-1} f(x_i)) = \frac{b-a}{N}(f(b)-f(a)) = \frac{\epsilon}{2}$ and the Riemann condition (p.448) is satisfied. The lower and upper sums take this form because f is increasing, so $\sup_{x\in[x_i,x_{i+1}]} f(x) = f(x_i)$ for all i and similarly for the inf's.

2 5

Consider $S_n = (-\frac{n}{2}, \frac{n}{2}) \times (-\frac{n}{2}, \frac{n}{2}) \times (-\frac{\epsilon}{n^2 2^{n+1}}, \frac{\epsilon}{n^2 2^{n+1}})$. Then $xy - plane \subset \bigcup_{n=1}^{\infty} S_n$. Also, $vol(S_n) = \frac{\epsilon}{2^n}$, so $\sum_{n=1}^{\infty} vol(S_n) = \epsilon$. Thus $m^*(xy - plane) = 0$ and consequently for any subset of the x-y plane is also measure zero.

3 7

Since f is continuous and $f(b) = -1$, there exists a δ such that $x > b - \delta \Rightarrow f(x) < -\frac{1}{2}$, so $\int_{b-\delta}^{b} f(x)dx < 0$. Since $\int_{a}^{b-\delta} f(x)dx + \int_{b-\delta}^{b} f(x)dx = \int_{a}^{b} f(x)dx = 0$, $\int_{a}^{b-\delta} f(x)dx > 0$ so there exists a $c \in (a, b-\delta)$ such that $f(c) > 0$. Then by the Intermediate Value Theorem, there exists a $d \in (c,b)$ such that $f(d) = 0$. Then by Rolle's Theorem, there exists a $t \in (a,d) \subset (a,b)$ such that $f'(t) = 0$.

4 11.

4.1 (a)

I shall use the indicator function formalism of p.451. In particular, $1_A \leq \sum_{i=1}^{N} 1_{A_i}$. This is obvious for $x \notin A$ (then $1_A(x) = 0$, and the RHS is non-negative), and if $x \in A$, then $x \in A_n$ for some n, so $1_A(x) = 1_{A_n}(x) \leq 1_{A_n}(x) + \sum_{i=1, i\neq n}^{N} 1_{A_i}(x)$, because again $\sum_{i=1, i\neq n}^{N} 1_{A_i}$ is non-negative. Thus:

$$vol(A) = \int 1_A \leq \int \sum_{n=1}^{N} 1_{A_n} = \sum_{n=1}^{N} \int 1_{A_n} = \sum_{n=1}^{N} vol(A_n)$$

4.2 (b)

4.2.1 (i)

If A has content zero, then for all $\epsilon > 0$ there exists a finite covering of A with rectangles of total volume $< \epsilon$, but that covering also satisfies the definition of measure zero on p.452, so we're done.

Say A is measure zero. Then there exists a countable covering of A with open rectangles R_i such that $\sum_{i=1}^{\infty} vol(R_i) < \epsilon$. Because A is compact, a finite subcovering R_{i_n} exists, and $\sum_{n=1}^{N} vol(R_{i_n}) < \sum_{i=1}^{\infty} vol(R_i) < \epsilon$ so A has content zero.

4.2.2 (ii)

B having volume is equivalent to $m^*(bd(B)) = 0$, so enough to show

$$m^*(bd(B)) = 0 \Leftrightarrow vol(bd(B)) = 0$$

Because B is bounded, $bd(B)$ is closed and bounded, so compact (we're implicitly in R^n). Then (i) applies and the two conditions are indeed equivalent.

5 15.

For the integral to be convergent, we need both $\int_0^1 x^p dx$ and $\int_1^{\infty} x^p dx$ to converge. Consider $p = 1$. Then:

$$\int_0^1 \frac{dx}{x} = \log 1 - (\lim_{x\to 0} \log x) \to \infty \text{ and } \int_1^{\infty} \frac{dx}{x} = (\lim_{x\to\infty} \log x) - \log 1 \to \infty$$

so the integral diverges. But for $p > -1$, $x^p > \frac{1}{x}$ for $x > 1$, so $\int_1^{\infty} x^p dx > \int_1^{\infty} \frac{dx}{x} \to \infty$, so the integral diverges for $p > -1$ also. For $p < -1$, $x^p > \frac{1}{x}$ for $x < 1$, so $\int_0^1 x^p dx > \int_0^1 \frac{dx}{x} \to \infty$, so the integral diverges for $p < -1$, too. Thus $\int_0^{\infty} x^p dx$ does not converge for any p.

6 21.

For $p > 1$, we have $x^{-p}|\sin x| \leq x^{-p}$, so $\int\limits_1^\infty x^{-p}|\sin x|dx \leq \int\limits_1^\infty x^{-p}dx = \frac{1}{1-p} - \lim_{x\to\infty} x^{1-p}$, which converges. For $p = 1$, Example 8.5.6./p.464 shows that the integral does not converge absolutely, and because $x^p|\sin x| \geq \frac{|\sin x|}{x}$ for $x \geq 1$, when $0 < p \leq 1$, the same is true for all $0 < p \leq 1$. It suffices to show that $\int\limits_1^\infty x^{-p}\sin x dx$ converges conditionally for these values of p. For that, perform integration by parts:

$$\int\limits_1^\infty x^{-p}\sin x dx = \cos 1 + \int\limits_1^\infty x^{-p-1}\cos x dx$$

which converges for the same reason as did $\int\limits_1^\infty x^{-p}|\sin x|dx$ (here we use $x^{-p-1}|\cos x| \leq x^{-p-1}$, and $\int\limits_1^\infty x^{-p-1}dx$ converges for $p > 0$). The proof is complete.

7 22.

Successive applications of L'Hopital's Rule show that $\lim_{x\to\infty} e^{-x}x^{p+1} = \lim_{x\to\infty} \frac{e^{-x}x^{p-1}}{(\frac{1}{x^2})} = 0$, so there is a d such that $x > d \Rightarrow \frac{e^{-x}x^{p-1}}{(\frac{1}{x^2})} < 1$, i.e. $e^{-x}x^{p-1} < \frac{1}{x^2}$, while for $x < d$, $e^{-x}x^{p-1} < x^{p-1}$. Then:

$$\int\limits_0^\infty e^{-x}x^{p-1}dx = \int\limits_0^d e^{-x}x^{p-1}dx + \int\limits_d^\infty e^{-x}x^{p-1}dx \leq \int\limits_0^d x^{p-1}dx + \int\limits_d^\infty \frac{dx}{x^2}$$

Both the latter integrals are p-type and converge for $p > 0$. Thus $\int\limits_1^\infty e^{-x}x^{p-1}dx$ converges for all p.

8 25.

See p.715 in the book.

9 29.

With $C = \bigcap\limits_{n=1}^{\infty} F_n$, where the F_n's are defined as on p.176, we observe that each F_n is a covering of C with rectangles of total volume $(\frac{2}{3})^n$, which goes to zero as $n \to \infty$. Note that this not only shows that C is measure zero, but also that it has content zero. This shouldn't be surprising, however, in light of Problem 11b (C is compact).