Problem Set 1 Solution Set

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1. *Exercise 1.2.1.* Define x_n inductively by $x_0 = 0$, $x_1 = \sqrt{2}$, $x_n = \sqrt{2 + x_{n-1}}$. Let $\lambda = \lim_{n\to\infty} x_n$.

   (a) Show that λ is a root of $\lambda^2 - \lambda - 2 = 0$.

   Solution. We know from example 1.2.10 that the above defined sequence converges. This means that we are allowed to take limits on both sides of the equation $x_n^2 = 2 + x_{n-1}$ to get $\lambda^2 = 2 + \lambda$. □

   (b) Find λ.

   Solution. The roots of $\lambda^2 - \lambda - 2 = 0$ are -1 and 2. But x_n is bounded below by 0 since it is monotonically increasing. So $\lambda = 2$. □

2. *Exercise 1.4.2.* Show that the sequence $x_n = e^{\sin(5n)}$ has a convergent subsequence.

   Solution. Since $|\sin(5n)| \leq 1$ and the exponential is an increasing function, we have $x_n \in [1/e, e]$. That is, the sequence is bounded. By the Bolzano–Weierstrass theorem, it must contain a convergent subsequence. □

3. *Exercise 1.4.5.* True or False: If x_n is a Cauchy sequence, then for n and m large enough, $d(x_{n+1}, x_{m+1}) \leq d(x_n, x_m)$.

   Solution. False. Many people had the right intuition for the problem, but didn't rigrously show that the proposition is false. All you need to do to disprove such statements is to come up with a counter-example. Many of you used the sequence $0, 1/2, 0, 1/4, 0, 1/8, 0, \ldots$. In other words

   $$x_n = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{2^{n/2}} & n \text{ even} \end{cases}.$$

   One easily shows this sequence converges to zero. Indeed, given $\epsilon > 0$, choose $N = 2\log_2(1/\epsilon) + 1$ so that

   $$|x_n - 0| \leq \epsilon \quad \text{whenever } n \geq N.$$

   Since the sequence convereges, it is Cauchy. However, no matter how large N is, one may always find distinct odd numbers $n, m \geq N$ such that

   $$d(x_n, x_m) = 0 < \left| \frac{1}{2^{(n+1)/2}} - \frac{1}{2^{(m+1)/2}} \right| = d(x_{n+1}, x_{m+1}).$$

$\square$

4. *End of Chapter Exercise 1.* For each of the following sets S find $\sup(S)$ and $\inf(S)$, if they exist:

   (a) $\{x \in \mathbb{R} \mid x^2 < 5\}$ We have $\sup(S) = +\sqrt{5}$ and $\inf(S) = -\sqrt{5}$.

   (b) $\{x \in \mathbb{R} \mid x^2 > 7\}$ We have $\sup(S) = +\infty$ and $\inf(S) = -\infty$.

   (c) $\{1/n \mid n \text{ a positive integer}\}$ We have $\sup(S) = 1$ and $\inf(S) = 0$.

   (d) $\{-1/n \mid n \text{ a positive integer}\}$ We have $\sup(S) = 0$ and $\inf(S) = -1$.

   (e) $\{.3, .33, .333, \dots\}$ We have $\sup(S) = 1/3$ and $\inf(S) = .3$.

5. *End of Chapter Exercise 4.* Show that $d = \inf(S)$ if and only if d is a lower bound for S and for any $\epsilon > 0$ there is an $x \in S$ such that $d \geq x - \epsilon$.

   Solution. ($\Longrightarrow$) Suppose that $d = \inf(S)$ but that for some $\epsilon > 0$ it is true that $d < x - \epsilon$ for all $x \in S$. Then for this ϵ, $d + \epsilon < x$ for all $x \in S$, i.e., $d + \epsilon$ is a lower bound for S that is strictly bigger than d. This is a contradiction.

   ($\Longleftarrow$) Now suppose that d is a lower bound for S and that for any $\epsilon > 0$ there is an $x \in S$ such that $d \geq x - \epsilon$. We must show $d = \inf(S)$. We know d is a lower bound for S by hypothesis. Now suppose c is another lower boud for S. We want $c \leq d$. Suppose $c > d$. Now choose $\epsilon = c - d$. Then there is an $x \in S$ such that $d \geq x - \epsilon$, or $d \geq x - (c - d)$, i.e., for this x, $c > x$. But this contradicts the assumption that c is a lower bound for S. $\square$

6. *End of Chapter Exercise 10.* Verify that the bounded metric $\rho(x, y) = d(x, y)/(1 + d(x, y))$ is indeed a metric.

   Solution. We verify the four conditions required for ρ to be a metric:

   - (positivity) $\rho(x, y) \geq 0$ for all x, y since $d(x, y) \geq 0$ for all x, y.
   - (non-degeneracy) $\rho(x, y) = 0 \Longleftrightarrow d(x, y) = 0 \Longleftrightarrow x = y$.
   - (symmetry) For all x, y we have

     $$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = \rho(y, x)$$

     since $d(x, y) = d(y, x)$ for all x, y.
   - (triangle inequality) Rewrite $\rho(x, y)$ as

     $$\rho(x, y) = \frac{d(x, y}{1 + d(x, y)} = 1 - \frac{1}{1 + d(x, y)}.$$

Since $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z, it follows that

$$\begin{aligned}
\rho(x,y) &\leq 1 - \frac{1}{1 + d(x,z) + d(z,y)} = \frac{d(x,z) + d(z,y)}{1 + d(x,z) + d(z,y)} \\
&= \frac{d(x,z)}{1 + d(x,z) + d(z,y)} + \frac{d(z,y)}{1 + d(x,z) + d(z,y)} \\
&\leq \frac{d(x,z)}{1 + d(x,z)} + \frac{d(z,y)}{1 + d(z,y)} \\
&= \rho(x,z) + \rho(z,y).
\end{aligned}$$

□

7. *End of Chapter Exercise 15.* Let x_n be a sequence in $\mathbb{R}$ such that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)/2$. Show that x_n is a Cauchy sequence.

   Solution. Note that

$$d(x_n, x_{n+1}) \leq \frac{d(x_{n-1}, x_n)}{2} \leq \frac{d(x_{n-2}, x_{n-1})}{2^2} \leq \cdots \leq \frac{d(x_1, x_2)}{2^{n-1}}.$$

   This can also be proved using mathematical induction. We want to show $\{x_n\}$ is Cauchy. Without loss of generality, assume $m \geq n$ and set $m = n+k$. By use of the triangle inequality for a general metric,

$$\begin{aligned}
d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\
&\leq \frac{d(x_1, x_2)}{2^{n-1}} + \cdots + \frac{d(x_1, x_2)}{2^{n+k-2}} \\
&= d(x_1, x_2)\left(\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^{n+k-2}}\right) \\
&\leq d(x_1, x_2) \cdot \sum_{i=n-1}^{\infty} \frac{1}{2^i} = \frac{d(x_1, x_2)}{2^{n-2}}.
\end{aligned}$$

   Notice that $d(x_1, x_2)$ is a constant, so given $\epsilon > 0$, it is always possible to choose N such that $d(x_1, x_2)/2^{N-2} \leq \epsilon$. For this N we obtain

$$d(x_n, x_m) \leq \frac{d(x_1, x_2)}{2^{n-2}} \leq \frac{d(x_1, x_2)}{2^{N-2}} < \epsilon$$

   whenever $n, m \geq N$, so $\{x_n\}$ is a Cauchy sequence. □

8. *End of Chapter Exercise 26.* Assume that $A = \{a_{m,n} \mid m = 1, 2, 3, \ldots \text{ and } n = 1, 2, 3, \ldots\}$ is a bounded set and that $a_{m,n} \geq a_{p,q}$ whenever $m \geq p$ and $n \geq q$. Show that

$$\lim_{n \to \infty} a_{n,n} = \sup A.$$

   Solution. We need the following lemma.

Lemma 0.1. *$d = \sup(S)$ if and only if d is an upper bound for S and for any $\epsilon > 0$ there is an $x \in S$ such that $d \leq x + \epsilon$.*

Proof. Imitate the proof of problem 5. □

Note that $\{a_{n,n}\}$ is a monotone increasing sequence bounded above, so it converges to some number which we will denote b. To show b is the supremum of A we must prove it is an upper bound for A and that for any $\epsilon > 0$ there is an $a_{p,q} \in A$ such that $b \leq a_{p,q} + \epsilon$.

Suppose b is not an upper bound for A. Then there is an element $a_{p,q} \in A$ such that $a_{p,q} > \lim_{n\to\infty} a_{n,n}$. Let $N = \max\{p, q\} + 1$. We get $a_{N,N} \geq a_{p,q} > \lim_{n\to\infty} a_{n,n}$, which is absurd since $\{a_{n,n}\}$ is a monotone increasing sequence. So b is an upper boud for A.

Now we show that for $\epsilon > 0$ there is an $a_{p,q} \in A$ such that $b \leq a_{p,q} + \epsilon$. For this $\epsilon > 0$ there is an N such that

$$|b - a_{n,n}| = b - a_{n,n} < \epsilon \quad \text{whenever } n \geq N.$$

Now choose p and q bigger than N to get $b < a_{p,q} + \epsilon$. This completes the proof. □

9. *End of Chapter Exercise 26.* Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy > 1\}$ and $B = \{d((x, y), (0, 0)) \mid (x, y) \in S\}$. Find $\inf(B)$.

   Solution. The answer is $\inf(B) = \sqrt{2}$. Since the question asked us to "find" this inf, I was not looking for an ultra rigurous justification of the answer. Many of you used calculus, which we have not formally developed yet, but this was fine.

   We want to minimize the function $f = \sqrt{x^2 + 1/x^2}$ which is equivalent to minimizing $g := f^2$. We compute

$$\frac{dg}{dx} = 2x - \frac{2}{x^3},$$

   Which is zero when $x = 1$ or when $x = -1$. Using the second derivative test we check that these are minima of the function and we compute $f(1) = f(-1) = \sqrt{2}$. □

10. *End of Chapter Exercise 29.* For any $x \in \mathbb{R}$ satisfying $x \geq 0$ prove the existence of $y \in \mathbb{R}$ such that $y^2 = x$.

    Solution. This exercise caused a lot of trouble, mainly because the proofs people came up with subtely assumed the existence of the square root of a number they would construct. But we are trying to show that square roots exist! :(

    Here is one way to approach the problem (See Bartle's *Elements of Real Analysis*). Let $S = \{u \in \mathbb{R} \mid u^2 \leq x\}$; this set is bounded above by x. Since S is not empty and is bounded above, it has a supremum. Call this supremum y. Either $y^2 < x$, $y^2 = x$ or $y^2 > x$.

    Suppose $y^2 < x$. Choose a natural number n such that $1/n < (x - y^2)/(2y + 1)$. Then

$$\begin{aligned}\left(y + \frac{1}{n}\right)^2 &= y^2 + \frac{2y}{n} + \frac{1}{n^2}\\ &\leq y^2 + \frac{2y+1}{n}\\ &< y^2 - (x - y^2) = x.\end{aligned}$$

Thus $(y+1/n) \in S$, and this is a contradiction since $y = \sup(S)$.

Now suppose $y^2 > x$. This time choose a natural number m such that $1/m < (y^2 - x)/(2y)$. Since $y = \sup(S)$, there exists $s_0 \in S$ such that $x - 1/m < s_0$. We conclude that

$$x < y^2 - \frac{2y}{m} < y^2 - \frac{2y}{m} + \frac{1}{m^2} = \left(y - \frac{1}{m}\right)^2 < s_0^2.$$

But then $s_0 \notin S$ since $s_0^2 > x$ and this is a contradiction.

Hence $y^2 = x$. □

11. *End of Chapter Exercise 32.*

(a) Give a reasonable definition for what $\lim_{n\to\infty} x_n = \infty$ should mean.

Solution. Given $A > 0$, there exists N such that

$$|x_n| > A \quad \text{whenever } n \geq N.$$

□

(b) Let $x_1 = 1$ and define inductively $x_{n+1} = (x_1 + \cdots + x_n)/2$. Prove that $x_n \to \infty$.

Solution. We can show, using induction, that $x_n = \frac{1}{2}\left(\frac{3}{2}\right)^{n-2}$ for $n > 1$.

We claim that $(3/2)^n > n$ for $n > 1$. We use induction. For the base case $3/2 > 1$. For the inductive step,

$$\left(\frac{3}{2}\right)^{n+1} > \frac{3n}{2} = n + \frac{n}{2} > n + 1.$$

Now we show that $x_n \to \infty$. Let $A > 0$ be given. Choose N large enough so that $(n-2)2 > A$ whenever $n > N$. Then

$$|x_n| = \frac{1}{2}\left(\frac{3}{2}\right)^{n-2} > \frac{n-2}{2} > A \quad \text{whenever } n \geq N.$$

□