Solutions
Real Analysis Midterm
Math 112 – Harvard University – Spring 2002

1. True or false: for any open set $A \subset \mathbb{R}$, we have $\text{int}(\overline{A}) = A$. Justify your answer.

   Answer: This is false. For example if $A = (0,1) \cup (1,2) \subset \mathbb{R}$, then $\overline{A} = [0,2]$ and $\text{int}(\overline{A}) = (0,2) \neq A$.

2. Let V be the vector space of all functions $f : [0,1] \to \mathbb{R}$, and define $\|f\| = |f(0)|$. Is $(V, \|\cdot\|)$ a normed vector space? Justify your answer.

   Answer: It is not. The function $f(x) = x$ is not equal to zero, but $\|f\| = 0$.

3. Let $A \subset \mathbb{R}$ be a non-compact set. Show there is an unbounded continuous function $f : A \to \mathbb{R}$.

   Answer: Since A is non-compact, it is either unbounded or not closed. If A is unbounded, then $f(x) = x$ is an unbounded function on A. If A is not closed, there is a sequence $a_n \in A$ converging to $b \notin A$, and then $f(x) = 1/(x-b)$ is an unbounded function on A.

4. Let x_n and y_n be sequences of real numbers such that $x_n \to x$ and $y_n \to y$. Prove that $x_n + y_n \to x + y$.

   Answer: Given $\epsilon > 0$, choose N such that $|x_n - x| < \epsilon/2$ and $|y_n - y| < \epsilon/2$ for all $n \geq N$. This is possible because $x_n \to x$ and $y_n \to y$. Then for $n \geq N$ we also have

   $$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \epsilon/2 + \epsilon/2 = \epsilon,$$

   and therefore $x_n + y_n$ converges to $x + y$.

5. Define a_n inductively by $a_1 = 1/2$ and $a_{n+1} = a_n + a_n^2$. Determine if a_n converges, and find its limit if it does.

   Answer: The sequence a_n diverges (it tends to infinity). In fact, since a_n^2 is positive, we have $a_{n+1} \geq a_n$; that is, the sequence is increasing. Thus $a_n \geq a_1 = 1/2$ for all n. But then we have

   $$a_{n+1} \geq a_n + a_n^2 \geq a_n + (1/2)^2$$

   for all n. In other words, a_n increases by at least $1/4$ each time n increases by 1. By induction we find $a_n \geq n/4 \to \infty$.