Real Analysis Final – Solutions

Math 112 – Harvard University – Spring 2002

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Prove that

$$f''(x) = \lim_{t\to 0} \frac{f(x+t) - 2f(x) + f(x-t)}{t^2}.$$

Proof 1. By Taylor's formula with remainder we have

$$f(x+t) = f(x) + tf'(x) + (t^2/2)f''(c_+)$$

where $|x - c_+| \le t$; and similarly for $f(x-t)$. Adding these formulas and subtracting $2f(x)$ gives

$$f(x+t) - 2f(x) + f(x-t) = (t^2/2)(f''(c_-) + f''(c_+)).$$

As $t \to 0$, c_+ and c_- converge to x. Since $f''(x)$ is continuous, the quotient of the expression above by t^2 converges to $f''(x)$. ■

Proof 2. Apply L'Hôpital's rule twice! ■

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$|f(x,y) + e^{x+2y} - y^2| \le e^{y^2} + |x|^{3/2} - 1$$

for all x and y. Show f is differentiable at the origin and compute $Df(0,0)$.

Proof. We claim $Df(0,0)$ exists and equals $(-1,-2)$. To see this, let $v = (x,y)$ and note that for $|v| \le 1$ the quantities $|y^2|$, $(e^{y^2} - 1)$ and $|x|^{3/2}$ are all $o(|v|)$. On the other hand, $e^{x+2y} = 1 + x + 2y + o(|v|)$ since e^{x+2y} is smooth. Thus for $|v| \le 1$ we have

$$|f(x,y) + 1 + x + 2y| = o(|v|),$$

which shows f is differentiable at $(0,0)$ and $Df(0,0) = (-1,-2)$. ■

3. (a) Give an example of a non-negative function $f : [0,1] \to \mathbb{R}$ such that f is unbounded on every open interval $(a,b) \subset [0,1]$.

(b) Is your function integrable, in the sense that the Riemann integral

$$\int_0^1 f = \lim_{M\to\infty} \int_0^1 \min(M, f(x))\, dx$$

exists and is finite? Justify your answer.

Answer. (a) (One possibility.) Let $(q_n)_1^\infty$ be an enumeration of the rational numbers in $[0,1]$, and define $f(x) = n$ if $x = q_n$ and $f(x) = 0$ otherwise. Since every open interval (a,b) contains infinitely many rational numbers, f is unbounded on every such interval.

(b) This function is not integrable. Once M is greater than one, $\min(M, f(x))$ is discontinuous at every point in $[0,1]$, so it is not Riemann integrable.

4. (a) Calculate the Fourier series $f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$ for the function $f(x) = \exp(2x)$ on $[0, 2\pi]$.

(b) Does the series converge uniformly on $[0, 2\pi]$?

(c) Give an example of a nonconstant function $f(x)$ whose Fourier series converges absolutely for all x.

Answer. (a) We have

$$\begin{aligned} a_n &= (2\pi)^{-1}\int_0^{2\pi} e^{2x}e^{-inx}\,dx = (2\pi)^{-1}\left.\frac{e^{(2-in)x}}{2-in}\right|_0^{2\pi} \\ &= \frac{e^{4\pi}-1}{2\pi(2-in)}. \end{aligned}$$

(b) The Fourier series does not converge uniformly since $f(0) \neq f(2\pi)$.

(c) The simplest example is $f(x) = e^{ix}$, whose Fourier series has only one nonzero term.

5. Find the linear function $p(x) = ax + b$ that best approximates $f(x) = x^3$ in $L^2[0,1]$, in the sense that $\|f - p\|^2 = \int_0^1 |f - p|^2$ is minimized.

Answer. The best approximation satisfies

$$\langle p, 1\rangle = \int_0^1 (ax + b)\,dx = a/2 + b = \langle f, 1\rangle = \int_0^1 x^3\,dx = 1/4,$$

as well as

$$\langle p, x\rangle = \int_0^1 (ax^2 + bx)\,dx = a/3 + b/2 = \langle f, x\rangle = \int_0^1 x^4\,dx = 1/5.$$

Solving these two equations gives $a = 9/10$ and $b = -1/5$. So the best linear approximation to $f(x)$ is $p(x) = (9/10)x - 1/5$.

Mark each assertion True (T) or False (F).

1. **True.** For any set $A \subset \mathbb{R}^2$, we have $\partial(\operatorname{int} A) \subset \partial A$.

2. **False.** Let $A_1, A_2, \ldots$ be compact subsets of $\mathbb{R}$ such that $A = \bigcup_1^\infty A_n$ is compact. Then $A = \bigcup_1^N A_n$ for some finite N.

   (E.g. Take $A_1 = \{0\}$, $A_n = \{1/n\}$ for $n > 1$.)

3. **False.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $f(g(x)) = x$ for all x. Then $g(f(x)) = x$ for all x.

   (E.g. take $n = 1$, $g(x) = e^x$, $f(x) = \log(x)$ for $x > 0$, $f(x) = x$ for $x \le 0$. Then $g(f(x)) = e^x$ for $x < 0$.)

4. **False.** The function $f(x, y) = -\sin(x) + x^2 + y^2$ has a local minimum at $(x, y) = (0, 0)$.

   ($Df(0, 0) = (-1, 0) \neq 0$.)

5. **False.** If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\nabla f(x)$ exists for all $x \in \mathbb{R}^n$, then $\nabla f(x)$ is continuous.

   (Try the case $n = 1$. The derivative can exist without being continuous.)

6. **False.** Let $p_n(x)$ be a sequence of cubic polynomials such that $p_n \to 0$ uniformly on a compact set $K \subset \mathbb{R}$. Then the coefficients of p_n also tend to zero.

   (E.g. if the compact set consists of just $x = 0$, only the constant term of p_n need tend to zero.)

7. **True.** Let $f : [0, 1) \to \mathbb{R}$ be a uniformly continuous function. Then $\lim_{x \nearrow 1} f(x)$ exists and is finite.

8. **False.** Suppose $\sum f_n(x)$ converges uniformly for $x \in [0, 1]$. Then $\sum f_n(0)$ converges absolutely.

   (E.g. take $f_n(x) = (-1)/n$.)

9. **False.** If $A \subset [0, 1]$ has measure zero, then A is a countable set.

   (E.g. the Cantor set is uncountable but has measure zero.)

10. **False.** Every bounded open set $A \subset \mathbb{R}$ has volume.

    (The boundary might have positive measure.)