Midterm Solutions

Math 101 — Sets, Groups and Knots

1. (10 points) List the elements of $A = \bigcap\{\{3,4,5\},5,6\}$.

   $A = \bigcap\{\{3,4,5\},\{0,1,2,3,4\},\{0,1,2,3,4,5\}\} = \{3,4\}$.

2. (10 points) Find a single generator for $G = \langle 34, 100\rangle \subset \mathbb{Z}$.

   $G = \langle \gcd(34,100)\rangle = \langle 2\rangle$.

3. (10 points) Write the complete statements of 3 of the 7 axioms of set theory, including the Axiom of Choice.

   Here are all the axioms:

   (1) $A = B$ iff $(x \in A \iff x \in B)$.
   (2) If A is a set and $P(x)$ is a logical proposition, then there is a set B such $x \in B$ iff $(x \in A$ and $P(x))$.
   (3) If A, B are sets then there is a set C such $x \in C$ iff $x = A$ or $x = B$.
   (4) If A is a set then there is a set U such that $x \in U$ iff $x \in B$ for some $B \in A$.
   (5) If A is a set then there is a set P such that $B \in P$ iff $B \subset A$.
   (6) There is a set A such that $\emptyset \in A$ and $x \in A \implies x \cup \{x\} \in A$.
   (7) For any set A there exists a function $c : \mathcal{P}(A) - \{\emptyset\} \to A$ such that $c(B) \in B$ for all nonempty sets $B \subset A$.

4. (10 points) Give a mathematical definition of the following sets:

   (a) $\bigcup A := \{x \;:\; x \in B \text{ for some } B \in A\}$.

   (b) $A \cap B := \{x \in A \;:\; x \in B\}$.

   (c) $3^A := \{f \in \mathcal{P}(A \times 3) \;:\; f \text{ is a function from } A \text{ to } 3\}$.

5. (15 points) Give an example of a group such that G is not cyclic, but every proper subgroup of G is cyclic. Justify your answer.

   A simple example is to take $G =$ the Klein 4 group $V = \{e,a,b,c\}$, where $a^2 = b^2 = c^2 = e$. Every proper subgroup has order 1 or 2, so it is cyclic. But V is not cyclic since it has no element of order 4.

   Another simple example is $G = S_3$ or D_3.

6. (15 points) (i) Give an example of a 1-1 map $f : \mathbb{N} \to \mathbb{N}$ that is not onto. (ii) Suppose $g : \mathbb{N} \to \mathbb{N}$ satisfies, for your chosen f, $g(f(a)) = a$ for all $a \in \mathbb{N}$. Prove that g is not 1-1.

(i) $f(n) = n + 1$ is $1 - 1$ but not onto, since $0 \notin f(\mathbb{N})$.

(ii) Let $b = g(0)$. Then $b = g(f(b)) = g(0)$, but $f(b) = b + 1 \neq 0$. Since g maps both $f(b)$ and 0 to the same point b, it is not 1-1.

(ii') (alternate solution): Since $g(f(a)) = a$, g is onto; thus if g is also 1-1, it is a bijection and $f = g^{-1}$. But then f is also onto, contrary to assumption.

7. (15 points) (i) Give examples of elements in S_5 with orders 3, 4, 5 and 6. (ii) Prove that A_5 contains no element of order 6.

   (i) $g = (123)$, (1234), (12345) and $(123)(45)$ have orders 3, 4 5 and 6 respectively.

   (ii) Suppose $g \in S_5$ has order 6, and let $H = \langle g \rangle$ be the cyclic group it generates. Since the orbits of g have size dividing $|H| = 6$, g is a product of cycles of lengths 1, 2 or 3. There must be at least cycles of order both 2 and 3, else g has order smaller than 6. Thus $g = (a_1a_2a_3)(a_4a_5)$ for some ordering $\{a_1, a_2, a_3, a_4, a_5\}$ of the elements $\{1, 2, 3, 4, 5\}$. But then $g = (a_1a_2)(a_2a_3)(a_4a_5)$ is an odd permutation, so $g \notin A_5$.

8. (20 points) Mark each of the following assertions True (T) or False (F).

   (a) F. Every function from a finite set to itself must be 1-to-1.

   (b) T. Every subgroup of an abelian group is abelian.

   (c) T. Every subgroup of a cyclic group is cyclic.

   (d) T. Every group of order 29 is cyclic. (Since 29 is prime.)

   (e) F. If A is a subset of B then A cannot be an element of B. (For example, $A = 2 = \{0, 1\}$ is both an element and a subset of $B = 3 = \{0, 1, 2\}$.

   (f) F. If A and B are groups, then either A is isomorphic to a subgroup of B or B is isomorphic to a subgroup of A. (For example, this is false for $A = \mathbb{Z}/3$ and $B = \mathbb{Z}/4$, since 3 does not divide 4.)

   (g) F. The group $\mathbb{Z}$ has no elements of finite order. (The identity element has order 1.)

   (h) F. Every infinite set satisfies $|A| \leq |\mathbb{R}|$. (The power set $A = \mathcal{P}(\mathbb{R})$ satisfies $|A| > |\mathbb{R}|$.)

   (i) F. The alternating group A_5 has order 120. (Its order is $5!/2 = 60$.)

   (j) T. If σ_1, σ_2 are odd permutations, then $\sigma_1\sigma_2$ is even.