Homework – Week 3

Sets, Maps and Knots

Math 101 – Harvard University – Fall 2005

Due Wednesday, 12 October 2005

1. Prove that for any function $f : A \to B$, we have
$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$$
for all subsets C, D of B. Show by an example that it is *not* true in general that $f(C \cap D) = f(C) \cap f(D)$ for all $C, D \subset A$.

2. Let $i, j \in \mathbb{N}$ be natural numbers. Considering i and j as sets, let
$$A = i - j = \{x \in i \ : \ x \notin j\}.$$
When is $A \in \mathbb{N}$? When A is a natural number, what number is it?

3. Which set has more elements, $\mathbb{N}^2$ or $2^{\mathbb{N}}$?

4. Prove that if A is infinite, there exists an injective map $f : \mathbb{N} \to A$. (Hint: you will need the Axiom of Choice.)

5. Prove that if A and B are finite sets, then $A \cup B$ is finite.

6. Prove the pigeon-hole principle: if A is finite, then every injective map $f : A \to A$ is surjective. (Hint: use induction on $|A| = n$).

7. Show that the map $f : \{0,1\}^{\mathbb{N}} \to [0,1]$ given by $f((a_i)) = \sum_0^\infty a_i/10^{i+1}$ is injective.

8. Find a surjective map $f : \{0,1\}^{\mathbb{N}} \to [0,1]$.

9. Find a bijective map $f : (0,1) \to \mathbb{R}$.

10. Prove that the two sets $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinality; that is, prove there is a bijection $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$.
(Hint: use the bijection $\mathcal{P}(\mathbb{N}) \leftrightarrow \{0,1\}^{\mathbb{N}}$ and the Schröder-Bernstein theorem.)