Homework 2

Sets, Maps and Knots

Due Monday, 3 Oct 2005

1. Show that for any subsets A, B and C of a set X, we have

$$X = B \cup (X - C) \cup (A \cap C) \cup ((X - A) \cap (X - B)).$$

2. Let A be a finite set, $|A| = n > 0$.
   (a) How many functions $f : A \to A$ are there?
   (b) How many of these are injective?

3. Let A be a set. Which functions $f : A \to A$ are *also* equivalence relations? (Remember that the function f is already a relation, namely $f = \{(a, f(a)) \ : \ a \in A\} \subset A \times A$.)

4. Let A, B and C be sets, and let $f : A \to B$ and $g : B \to C$ be functions.
   (a) Prove $g \circ f : A \to C$ is injective if f and g are injective.
   (b) Discuss the assertion "$g \circ f$ is surjective iff f and g are surjective."
   (c) Suppose $h : B \to C$ is a function, and we know $g \circ f = h \circ f$. What natural condition on f allows us to conclude that $g = h$?
   (d) Prove your assertion.

5. Show for any set A, we have $A = \bigcup \mathcal{P}(A)$. Given an example of a set A with $A \neq \mathcal{P}(\bigcup A)$.

6. Let A be a set. List the elements of $A^{\emptyset}$, $\emptyset^{\emptyset}$ and $\emptyset^{\{A\}}$. (Recall that X^Y denotes the set of all functions from Y to X.)

7. Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(x) = -2x$ for $x \leq 0$ and $f(x) = 2x - 1$ for $x > 0$. Prove that f is a bijection.

8. Construct a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

   (There are many possible answers. Here is a hint for one of them: every integer $n > 0$ can be factored as $n = 2^a(2b + 1)$, a power of two times an odd number.)