Final Solutions
Sets, Groups and Knots
Math 101 – Harvard University

1. Let $E_i \subset \mathcal{P}(\mathbb{Z})$ be the collection of all subsets $A \subset \mathbb{Z}$ such that $|A| = i$. Show there is a bijection between E_2 and E_3.

   Answer. There is an injective map $E_2 \to \mathbb{Z}^2$ sending $\{a, b\}$ to the ordered pair (a, b) with $a < b$; and similarly an injective map $E_3 \to \mathbb{Z}^3$ sending $\{a, b, c\}$ to (a, b, c) with $a < b < c$. Thus E_2 are E_3 are both countable; and clearly they are both infinite. By the Schröder-Bernstein theorem, any two countable infinite sets have the same cardinality, so there is a bijection $f : E_2 \to E_3$.

2. Prove there is a set $E \subset \mathbb{R}$ such that every real number x can be written uniquely as $x = e + q$ with $e \in E$ and $q \in \mathbb{Q}$.

   Answer. Define a relation on $\mathbb{R}$ by $x \sim x'$ if $x - x' \in \mathbb{Q}$. Since $\mathbb{Q}$ is a group, $\sim$ is an equivalence relation. By the Axiom of Choice, we can choose a single element from each equivalence class, obtaining a set E. Then for any $x \in \mathbb{R}$, there is a unique $e \in E$ such that $[x] = [e]$, or equivalently such that $x - e = q \in \mathbb{Q}$, yielding the unique expression $x = e + q$.

3. Show that any two elements of order 5 in S_9 are conjugate (that is, $x = gyg^{-1}$ for some $g \in S_9$). Show this fails for S_{10}.

   Proof: Any element of order 5 in S_9 must be a single cycle of order 5, and any two cycles of the same order are conjugate. (If $x = (a_1 \ldots a_n)$ and $y = (b_1 \ldots b_n)$ then $y = gxg^{-1}$ for any such that $g(b_i) = a_i$.) On the other hand, in S_{10} the elements $x = (12345)$ and $y = (12345)(6789(10))$ are both of order 5, but not conjugate.

4. a) Find all the elements of order two in D_n. (You can express elements of D_n as $r^i f^j$.) (b) Determine when 2 such elements are conjugate. (c) How many 'types' of elements of order two does D_n contain, if we regard conjugate elements as being of the same type?

   Answer. Note that $D_n = \langle f, r \,:\, r^n = f^2 = e, rf = fr^{-1}\rangle$ has order $2n$ (we assume $n > 0$). (a) The elements of order two are those of the form $r^i f$ and, when n is even, $r^{n/2}$. (b) Noting that $r(r^i f)r^{-1} = r^{i+2} f$, we see that all these elements are conjugate when n is odd. When n is even, $r^i f$ is conjugate to $r^j f$ iff $i \equiv j \bmod 2$, and $r^{n/2}$ is in a conjugacy class by itself. (c) There are 3 types when n is even, and one type when n is odd.

5. Prove that any group G of order 34 contains an element of order 17.

   Answer. Every element of G has order 1, 2, 17 or 34, since the order an element divides the order of G. If G has no element of order 17, then it also has no element x of order 34 (since x^2 would have order 17), so every element of G has order 1 or 2.

   Now if $x^2 = e$ for every element in G, then G is be abelian (since $ab = ab(ba)^2 = ab^2aba = a^2ba = ba$). By the classification of abelian groups, if G is abelian then $G \cong \mathbb{Z}/17 \times \mathbb{Z}/2$ and so G has an element of order 17, namely $(1, 0)$.

6. Let L be the 3-component link $L = 6^3_2$ from Adams' table. (a) Compute a presentation for the fundamental group $G(L)$. (b) Find a surjective homomorphism $\phi : G(L) \to \mathbb{Z}^3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

   Answer. (a) $G(L) = \langle a, b, c, d, e, f \ : \ ea = ab, fb = bc, dc = ca, ba = db, db = ed, ec = fe\rangle$. (b) Set $\phi(a) = \phi(d) = (1, 0, 0)$, $\phi(b) = \phi(e) = (0, 1, 0)$ and $\phi(c) = \phi(f) = (0, 0, 1)$. The relations are easily checked.

7. Prove that $G(L)$ is nonabelian.

   Answer. L contains the unlink on two components, which can be tricolored, so $G(L)$ admits a surjective homomorphism to S_3. Explicitly, set $\phi(a) = \phi(d) = \mathrm{id}$, $\phi(b) = \phi(e) = (23)$ and $\phi(c) = \phi(f) = (31)$. The relations are easily checked, so we obtain a homomorphism onto S_3. Since S_3 is nonabelian, so is $G(L)$.

8. (a) Show that the knot 6_3 is equivalent to its mirror image. (b) Explain why no other knot of 6 or 7 crossings in Adam's table is equivalent to its mirror image.

   Answer. (a) For 6_3 the equivalence can be proved using a sequence of Reidemeister moves. (b) If a knot is equivalent to its mirror image, then the coefficients of its Jones polynomial must be palindromic (they read the same forwards and backwards). Looking at the coefficients given in Adams' table, we see the only knots with symmetric polynomials on 7 crossings or less are 4_1 and 6_3.

9. Let K be the knot projection 5_1 in Adams' table. Compute $w(K)$, $\langle K\rangle$, $X(K)$ and $V(K)$.

Answer. Resolving two crossings, we find:

$$\begin{aligned}\langle 5_1\rangle &= A\langle\text{unknot of writhe 4}\rangle + A^{-1}\langle\text{4-crossing link}\rangle\\ &= A(-A^3)^4 + A^{-1}A\langle\text{unknot of writhe 3}\rangle + A^{-2}\langle 3_1^-\rangle\\ &= A^{13} + (-A^3)^3 + A^{-2}(-A^{-5} - A^3 + A^7)\\ &= -A^{-7} - A + A^5 - A^9 + A^{13}.\end{aligned}$$

All 5 crossings in 5_1 are unsafe, so $w(K) = -5$. Therefore

$$X(K) = (-A^3)^5\langle K\rangle = A^8 + A^{16} - A^{20} + A^{24} - A^{28}.$$

Finally we substitute $A = t^{-1/4}$ to obtain:

$$V(K) = -t^{-7} + t^{-6} - t^{-5} + t^{-4} + t^{-2}.$$

10. Compute the Jones polynomial of the knot shown below.

Answer. This is the trefoil knot 3_1^-! So $V(K) = -t^{-4} + t^{-3} + t^{-1}$.