Throughout, $X$ is a compact Riemann surface.

1. Prove that $H^2(X, \mathcal{O}) = 0$.

2. Let $D = D_+ - D_-$ where $D_\pm$ are effective divisors with disjoint support. Prove (by elementary means) that $\dim \mathcal{O}_D(X) \leq \deg(D_+) + 1$.

3. Let $V \subset \mathcal{M}(X)$ be a finite-dimensional subspace of positive dimension. Show there is a single divisor $D \geq 0$ such that $(f)_\infty = D$ for most $f \in V$; in particular, most $f \in V$ have the same degree. Describe the set of $f \in V$ such that $(f)_\infty \neq D$.

(Here $(f)_\infty = \sum n_P \cdot P \geq 0$ is the polar divisor of $f$; $P$ ranges over the poles of $f$, and $n_P = -\text{ord}_P(f)$.)

4. Let $X$ be a compact Riemann surface of topological genus $g$, and assume $f \in \mathcal{M}(X)$ is a nonconstant meromorphic function. Use the Riemann-Hurwitz formula to show $\deg(df) = 2g - 2$.

5. Let Div be the sheaf of divisors on $X$; that is, Div$(U)$ is the group of all formal sums $\sum_{P \in U} a_P P$, $a_P \in \mathbb{Z}$ such that $\{P : a_P \neq 0\}$ is a discrete subset of $U$. There is a natural map $\mathcal{M}^*(U) \to \text{Div}(U)$ given by $f \mapsto (f)$.

(i) Show there sequence of sheaves $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \text{Div} \to 0$ is exact.

(ii) Show that $H^1(X, \text{Div}) = 0$.

6. Let $X = \hat{\mathbb{C}}$ with the covering $\mathcal{U} = \{U_1, U_2\} = \{\mathbb{C}, \hat{\mathbb{C}} - \{0\}\}$, and let $\Omega$ be the sheaf of holomorphic differentials. Prove that $H^1(\mathcal{U}, \Omega) \cong \mathbb{C}$.

7. Let $H^2(\Delta)$ be the space of holomorphic functions on the unit disk such that $\|f\|_2^2 = \int_\Delta |f|^2$ is finite. Prove that $H^2(\Delta)$ is a Hilbert space, and that for any $r < 1$, the map $T : H^2(\Delta) \to H^2(\Delta)$ given by $(Tf)(z) = f(rz)$ is a compact operator.