Advanced Complex Analysis: Homework 5

1. Let \( P = \{2, 3, 5, \ldots\} \) be the prime numbers, endowed with the cofinite topology (a set is closed iff it is finite or the whole space). Given an open set \( U = P - \{p_1, \ldots, p_n\} \), let \( \mathcal{F}(U) \) be the ring \( \mathbb{Z}[1/p_1, \ldots, 1/p_n] \subset \mathbb{R} \), and let \( \mathcal{F}(\emptyset) = (0) \).

Show that with the natural restriction maps, \( \mathcal{F} \) is a sheaf of rings over \( P \). What is the stalk \( \mathcal{F}_p \)?

2. Let \( p(x, y) = \sum_{0 \leq i+j \leq d} a_{ij}x^i y^j \in \mathbb{C}[x, y] \) be a polynomial defining a smooth Riemann surface \( X^* \subset \mathbb{C}^2 \) of degree \( d \). Let \( \pi : X^* \to \mathbb{C} \) be projection to the \( x \)-coordinate.

(i) Show that for ‘typical’ \( p \) (i.e. for generic coefficients \( a_{ij} \)), the map \( \pi \) is proper of degree \( d \).

(ii) Determine the ‘typical’ number of critical points \( |C(\pi)| \).

(iii) Let \( \overline{\pi} : X \to \hat{\mathbb{C}} \) denote the compact branched covering obtained by completing \( \pi : X^* \to \mathbb{C} \). Show that \( \pi^{-1}(\infty) \) ‘typically’ consists of \( d \) points.

(iv) Derive a formula for the genus of \( X \) in terms of \( d \).

3. (i) Prove that for any sheaf of abelian groups \( \mathcal{F} \) on a space \( X \) and any open set \( U \subset X \), the group \( \mathcal{F}(U) \) is naturally isomorphic to the group of sections of the espace étalé \( |\mathcal{F}| \) over \( U \).

(ii) Give an example of a presheaf on a space \( X \) such that \( \mathcal{F}(X) \cong \mathbb{Z} \) but \( \mathcal{F}_x = (0) \) for every \( x \in X \).

(iii) Let \( \mathcal{F} \) be the sheaf of locally constant functions on \( X \) with values in a fixed abelian group \( G \). Give an explicit description of \( |\mathcal{F}| \) (including its topology).

4. Let \( U \subset \mathbb{C} \) be a connect open set containing \( z = 0 \). Show there is a power series \( f(z) = \sum a_n z^n \), convergent near \( z = 0 \), which can be analytically continued to \( U \) but to no larger Riemann surface.

5. Given the first 3 nonzero terms in the solution to \( P(T) = T^5 + zT + z = 0 \) near \( z = 0 \) by a Puiseux series.

6. Let \( U \subset \mathbb{C} \) be an open set. Prove that every complex-linear ring homomorphism \( \chi : \mathcal{O}(U) \to \mathbb{C} \) is given by \( \chi(f) = f(p) \) for some \( p \in U \).