1. (i) Show there is a subgroup $\Gamma \subset \text{Aut}(\hat{\mathbb{C}})$ isomorphic to $S_4$. (Hint: consider the symmetries of the cube.)

(ii) Find a rational map $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that is a degree 24 regular branched covering with deck group $S_4 \cong \Gamma \subset \text{Aut}(\hat{\mathbb{C}})$. Show that $f$ can be expressed as the composition of two rational maps of degree 4 and 6. (Hint: $S_3$ is a quotient of $S_4$.)

2. Let $X$ be a compact Riemann surface of genus $g$ and let $A \subset X$ be a finite set with $|A| = n$ even.

(a) Show there exists a compact Riemann surface $Y$ and a holomorphic map $f: Y \to X$ of degree 2, branched exactly over $A$.

(b) What is the genus of $Y$ (in terms of $n$ and $g$)?

(c) Draw a (topological) picture of the map $f: Y \to X$ in the case $(g, n) = (1, 2)$.

(d) In terms of $(g, n)$, how many possibilities are there for $f: Y \to X$, up to isomorphism over $X$? (Two maps $f_i: Y_i \to X$, $i = 1, 2$ are isomorphic over $X$ if there is an isomorphism $g: Y_1 \to Y_2$ such that $f_2 \circ g = f_1$.)

3. Find a Belyi map $f: X \to \hat{\mathbb{C}}$ where $X = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$ is the square torus. That is, find a holomorphic map from $X$ to the Riemann sphere branched over just $\{0, 1, \infty\}$.

4. Show that for any finite group $G$, there exists a regular covering map $f: X \to Y$ between compact Riemann surfaces such that $\text{Deck}(X/Y) \cong G$.

If $G = (\mathbb{Z}/2)^n$, what is the smallest possible genus for $Y$?

5. Show that for each $d \geq 1$, there exists a complex torus $X \cong \mathbb{C}/\Lambda$ and an analytic map $f: X \to X$ of degree $d$. 

Advanced Complex Analysis: Homework 4