This final homework is to be completed without collaboration.

All work should be your own. Due Wednesday, May 9th, by 11 am, in Natasha’s mailbox. (Late homework cannot be accepted.)

Throughout $X$ is a compact Riemann surface.

1. Prove that if the genus of $X$ is $3$ or more, then the linear system $|2K|$ provides an embedding of $X$ into $\mathbb{P}^{3g-4}$. What happens when $g = 2$?

2. Find the flexes of the Fermat quartic $X$ defined by $x^4 + y^4 = 1$. Are there $(g-1)g(g+1) = 24$ of them? Explain. Give explicitly a meromorphic function $f(x, y)$ on $X$ with a single pole of order 3.

3. Suppose $X$ has genus two. Give an example of a group automorphism $h : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ that is not induced by a homeomorphism $H : X \to X$.

4. Let $X$ be the Riemann surface of genus two obtained as a 2-fold branched covering of $\hat{\mathbb{C}}$ branched over $0, \infty$ and the 4th roots of unity. Find an explicit lattice $\Lambda \subset \mathbb{C}^2$ such that $\text{Jac}(X) \cong \mathbb{C}^2/\Lambda$.

5. Let $\mathcal{L}$ be the sheaf of holomorphic sections of a holomorphic line bundle $L \to X$. Suppose $H^1(X, \mathcal{L})$ is finite. Show that for any $P \in X$, there is a meromorphic section $f : X \to L$ which is holomorphic outside $P$.

6. Show that the space $X^{(n)} = X^n/S_n$ is the space of unordered $n$-tuples of points on $X$, carries the natural structure of a complex $n$-manifold. (Hint: there is an isomorphism between $\mathbb{C}^{(n)}$ and $\mathbb{C}^n$ sending $(r_1, \ldots, r_n)$ to the coefficients of the polynomial $(z - r_1) \cdots (z - r_n)$.)

7. Fix a divisor $D$ of degree $n > 0$ on a $X$. Define $\phi : X^{(n)} \to \text{Jac}(X)$ by $\phi(P_1, \ldots, P_n) = [-D + \sum P_i]$. Show for each $x \in \text{Jac}(X)$, fiber $\phi^{-1}(x)$ is a projective space $\mathbb{P}^k$ (or empty). Express $k$ in terms of $(P_i)$ when $x = \phi(P_i)$. 