Advanced Complex Analysis: Homework 11

Throughout $X$ is a compact Riemann surface.

1. Let $D$ be a divisor on $X$ such that $h^0(D) > 0$ but $|D|$ is not base-point free. Describe the hyperplane sections of the natural map $\phi_D : X \to \mathbb{P}H^0(X, \mathcal{O}_D)^*$. (Hint: consider the largest divisor $B$ such that $B \leq E$ for all $E \in |D|$.)

2. Let $X \subset \mathbb{P}^3$ be a canonical curve of genus 4 (this means $X$ is embedded by the complete linear system $|K|$; in particular it is a smooth curve of degree $2g - 2 = 6$.) Suppose 3 distinct points $P_1, P_2, P_3 \in X$ lie on a line $L$. Show that for any hyperplane $H$ containing $L$, the other 3 points of $H \cap X$ also lie on a line.

3. Let $X \subset \mathbb{P}^2$ be a smooth curve of degree $d = 2$. Do all the automorphisms of $X$ extend to automorphisms of $\mathbb{P}^2$? What happens if $d = 3$ or $d = 4$?

4. Suppose $g(X) = 1$. Prove that for any 3 points $P_i \in X$, there exists an embedding $f : X \to \mathbb{P}^2$ such that $D = P_1 + P_2 + P_3$ is a hyperplane section of $f(X)$.

5. Let $X$ be a hyperelliptic Riemann surface of genus $g \geq 2$, with $f : X \to \mathbb{P}^1$ a holomorphic map of degree 2.
   (i) Prove that $h^0(gP) \geq 2$ iff $P$ is one of the critical points of $f$.
   (ii) Prove that any other degree 2 holomorphic map $h : X \to \mathbb{P}^1$ has the form $h(z) = (af(z) + b)/(cf(z) + d)$.

6. Construct explicitly a holomorphic map $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ exhibiting the projective line as the normalization of the nodal cubic curve defined by $y^2 = x^2(x - 1)$.

7. Let $p(x)$ be a monic polynomial of degree $d$ with distinct roots. Is the curve in $\mathbb{P}^2$ defined by $y^2 = p(x)$ smooth?