

Tutorial: Sphere Packings

Abhinav Kumar

January 31, 2006

http://math.harvard.edu/~abhinav/tutorial_sp/

1 Introduction

This tutorial will focus on sphere packing and related problems. The basic sphere packing problem in \mathbb{R}^n is to find the densest collection of nonoverlapping spheres of equal size (i.e. occupying the largest fraction of the volume). It's an extremely hard problem in general, but there are many ingenious methods used to obtain upper and lower bounds for the maximum possible density, and to try to understand the variation with n of this density. Some related problems are:

- The kissing problem: How many nonoverlapping spheres of the same size can you pack touching a single sphere)
- The lattice packing problem (the spheres are centered at the points of a lattice in \mathbb{R}^n), which also has connections with quadratic forms in number theory.
- The covering problem: What's the most efficient covering of \mathbb{R}^n by overlapping spheres, i.e. such that the fraction of space wasted by overlapping is minimum.
- Spherical codes: This is sphere packing on a sphere! How many points can you put on a unit sphere so that the spherical distance between any two is at least some given ϕ ?

We will talk about the techniques for tackling these problems, and look at some concrete answers in low dimensions. There are no prerequisites, except for an interest in geometry and some facility in analysis.

2 Syllabus

Basic setup: sphere packings, lattice and non-lattice packings, results in low dimensions. The kissing and covering problems. Some classic examples (Kepler's conjecture, Gregory-Newton problem).

Minkowski-Hlawka lower bound. Upper bounds of Blichfeldt, Rogers.

Orthogonal polynomials. Poisson summation and Cohn-Elkies upper bounds.

Constructions of sphere packings from codes.

Lattices and theta functions. Extremal lattices and Voronoi's theorem.

Spherical codes. Jacobi polynomials and the linear programming bound. Optimal kissing numbers in 8 and 24 dimensions.

Kabatiansky-Levenshtein upper bounds on kissing numbers and sphere packing density.

Polytopes. Voronoi and Delaunay cells.

The sphere covering problem. Upper and lower bounds. Lattice coverings, results in low dimensions.

3 References

- Conway and Sloane, *Sphere packings, lattices and groups*.
- Zong, *Sphere packings*

4 Possible final paper topics

1. Minkowski-Siegel mass formula
2. The kissing number problem in 3 or 4 dimensions.
3. Computer experiments for finding good spherical codes.
4. Spherical designs
5. Linear programming bounds in coding theory.