MODULAR MAGIC

The Theory of Modular Forms and the Sphere Packing Problem in Dimensions 8 and 24

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Contents

Preface

This thesis will give a motivated exposition of Viazovska's proof that the E_8 lattice packing is the densest sphere packing in dimension 8, as well as an overview of the (very similar) proof that the Leech lattice is optimal in dimension 24. In chapter 1, we give a brief history of the sphere packing problem, discuss some of the basic definitions and general theorems concerning sphere packing, and offer constructions of the E_8 and Leech lattices. We do not, however, delve deeply into the miraculous properties of these lattices. In chapter 2 we give a more or less complete introduction to the theory of modular forms as it will be used in Viazovska's proof. For those already familiar with this material, it is recommended to begin with chapter 3. Chapter 3 contains a proof of the Cohn-Elkies linear programming bound, its application to the trivial 1-dimensional sphere packing problem, and describes some of the numerical results that can be computed from the Cohn-Elkies bound. Chapter 4 gives a lengthy discussion of the motivation for Viazovska's magic function construction and then a rigorous proof that it works as needed. This chapter contains the main content of this thesis. Chapter 5 gives a concise overview of the corresponding magic function construction in dimension 24. Finally, chapter 6 describes some further questions concerning magic functions, the Cohn-Elkies bound, sphere packing, Fourier Interpolation and the 8 and 24 dimensional proofs.

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1. Background and History

1.1. **Some History.** How does one pack congruent spheres as densely as possible? This seemingly innocent question – one that anyone who has tried to pack oranges or stack marbles is already intimately familiar with – turns out to be one of the most tantalizingly difficult problems in geometry. Spheres, unlike cubes, do not fit together well, and the kinds of arrangements one can create by trying to pack spheres (and their multidimensional analogues) together are diverse and interesting. Moreover, the theory of sphere packings has fascinating connections to other areas of mathematics: quadratic forms, number theory, lattices, finite groups, and modular forms, to name a few.

The central question in the study of sphere packing is to find the largest proportion of d-dimensional Euclidean space that can be covered by non-overlapping unit balls. For general d, the answer remains unknown; in fact, we do not even know whether the maximally dense configurations are lattice packings or even periodic for most d (indeed, heuristic evidence suggests that optimal packings will likely *not* be lattice packings for most d [8].

The case of sphere packing in everyday 3-dimensional space was of great historical importance. It was conjectured by astronomer and mathematician Johannes Kepler, in his 1611 paper *Strena seu de nive sexangula* ('On the Six-Cornered Snowflake'), that the densest packing of spheres in 3 dimensions is the so-called face-centered-cubic (FCC) lattice, which covers $\pi/\sqrt{18}$ or about 74.05% of 3-dimensional space [21]. The background for Kepler's interest in sphere packing is itself an interesting historical footnote. Kepler was inspired to study sphere packings by fellow astronomer Thomas Harriot (1560–1621), who was a friend and assistant of the famous explorer Sir Walter Raleigh (1554–1618), the early New World colonist whose doomed 'Roanoke' expedition has been the subject of much mystery. It was Raleigh who had set Harriot the (very practical) problem of finding the most efficient way to stack cannonballs on his ships [36]. One might say that the modern theory of sphere packing and United States of America share a common ancestor.

Little progress was made on the so-called 'Kepler conjecture' until Gauss [16], in 1831, who proved that Kepler's packing achieves maximal density among *lattice* packings of spheres (that is, Gauss showed that we insist upon the centers of the spheres forming a lattice in $\mathbb{R}^3$, then the FCC is the densest). The question of irregular packings turned out to be of great difficulty – indeed, over restricted subsets of $\mathbb{R}^3$ it is possible to find irregular packings that achieve density greater than Kepler's – but all known examples of these failed to extend to all of $\mathbb{R}^3$.

In 1953, László Fejes Tóth (1915-2005), one of the progenitors of discrete geometry (and the theory of sphere packings specifically), demonstrated that, in principle, one could reduce the problem of irregular packings in Kepler's conjecture to verifying a finite (but exceedingly large) set of computations; Fejes Tóth himself observed that a computer could in theory verify all the necessary cases, though at the time, the technology was insufficient to make this method practicable [14].

Thomas Hales, a number theorist then at the University of Michigan, was the first to implement Fejes Tóth's program. Hales and his student Samuel Ferguson completed this computer-assisted proof by 1998, depending upon the computer-assisted resolution of around 100,000 linear programming problems [18]. This proof was as interesting sociologically as it was mathematically – it was the second significant mathematical result to be resolved through exhaustive computer-assisted brute force (the first being the famous four-color theorem, proved in 1976 by Appel and Haken). It represents one of the first computer-age proofs: an argument that no human knows in full detail, but which establishes the result with complete certainty – perhaps even greater certainty than many proofs which are not computer-assisted, and so liable to human fallibility. Recently, Hales and collaborators published on the ArXiv a computer-verification of the proof of the Kepler conjecture [19]. It was accepted for publication in the journal *Forum of Mathematics* in 2017.

A parallel – though less protracted – story transpired for $d = 2$, the problem of packing congruent circles as densely is possible in $\mathbb{R}^2$. Here the hexagonal (honeycomb) circle-packing is optimal, covering $\pi/\sqrt{12}$ or about 90.69% of $\mathbb{R}^2$. Joseph-Louis Lagrange (1736-1813) played the analogous role to Gauss in this story, proving that the honeycomb lattice is optimal among lattice circle packings in $\mathbb{R}^2$; the first proof that the hexagonal packing is optimal among all packings came with Axel Thue (1863-1922) in 1890 [37]. Yet, while this result is often called "Thue's theorem," there is some contention over whether Thue's proof is fully correct. The first proof universally acknowledged to be complete was given by Fejes Tóth (1940) [15].

This thesis concerns the remarkable recent results which resolve the sphere packing problem in dimensions 8 and 24. It was proved as early as 1935 that the E_8 lattice gives the densest possible packing among *lattice* sphere packings in 8 dimensions [2]. However, the proof that the Leech lattice is the densest among lattice packing in 24 dimensions came as late as 2004 [7]. The 8-dimensional problem for general, possibly irregular packings was solved by Maryna Viazovska in 2016 [38], and her method was quickly re-purposed to solve the general sphere packing problem in 24 dimensions [8]. Viazovska's solution consisted of finding an optimal function for the so-called "linear programming bound" of Cohn-Elkies, which was in turn inspired by similar "linear programming bounds" for kissing numbers and error-correcting codes. We shall give a brief history of these background developments.

A closely related issue to the sphere packing problem, the so-called "kissing number problem" acted as an early incubator for the methods and ideas employed in resolving the sphere packing problem in $d = 8$ and $d = 24$. The kissing number in dimension n is defined to be the maximal number of non-overlapping, congruent spheres that can be placed around a single, central, sphere of the same radius. (The shell of spheres thus "kiss" the sphere in the middle). In two dimensions, it is fairly clear that one can place six pennies around a single penny but no more. In 1694, the corresponding question in three dimensions became the subject of a famous debate between Isaac Newton (1643-1727) and David Gregory (1661-1708). Newton believed that only twelve spheres could fit, whereas Gregory believed that a thirteenth could be squeezed in. Newton was ultimately vindicated, though the first rigorous proof was given in 1953 by Kurt Schütte (1909-1998) and Bartel van der Waerden (1903-1996) [34]. Notably, the "fit" of the 12 spheres around the single central sphere (unlike in two dimensions) is very poor – one can, in fact, continuously roll the 12 spheres on the central sphere without ever lifting the spheres off of the central sphere, and, eventually, swap the positions of any two spheres.

In the beginning of the 1970's Phillipe Delsarte made a major breakthrough in the closely related theory of binary codes, where he applied the theory of linear programming to establish strong upper bounds for cardinalities of error-correcting codes [11]. Delsarte, Goethals, and Seidel soon applied this theory to the case of so-called "spherical codes" (which correspond to finite sets of points on a sphere in some dimension) to give very good upper bounds on kissing numbers [12]. The technique involves constructing a certain auxiliary function with specified properties; given such a function, one can find a corresponding bound on kissing numbers. In the case of dimensions 8, and 24, it was found to be possible force these upper bounds to actually equal the kissing numbers coming from known lattice packings (namely E_8 in dimension 8; and Λ_{24}, or the Leech lattice, in dimension 24). With this technique Neil Sloane (1939–) and Andrew Odlyzko (1949–) [30] and, independently, Vladimir Levenshtein (1935–2017) [22], proved that the kissing numbers in dimension 8 and 24 are 240 and 196560, respectively; with these bounds attained by the first shell of the E_8 and the Leech lattice sphere packings, respectively.

Interestingly, while the Delsarte technique itself was insufficient to solve the kissing number problem in 4 dimensions, it turned out that the method could be cleverly modified to resolve this case as well. It was known that one could fit 24 hyperspheres around a single congruent hypersphere (this is accomplished by the first shell of the D_4 lattice packing, which is also conjectured to be the best sphere packing in dimension 4). However, Delsarte's method in four dimensions proved only able to give an upper bound of 25. Oleg Musin found a rather subtle modification of the Delsarte Method to reduce the bound to 24, thus solving the 4-dimensional kissing number problem [28; 27]. Interestingly, Musin then used this technique to offer an alternative proof that the 3-dimensional kissing number is 12 [29].

In 2003 Henry Cohn and Noam Elkies introduced the technique of linear programming bounds for sphere packings in analogy with the linear programming bounds used in the theory of error-correcting codes and kissing numbers [6]. Like the Delsarte bound, the Cohn-Elkies bound depends on finding some auxiliary function subject to certain constraints. Efforts to optimize the this function revealed that, like with kissing numbers, the Cohn-Elkies bound seemed to actually give the optimal packing density in dimensions 8 and 24. That is, the upper bounds coming from well-chosen functions f could be made to be extremely close to the densities of the lattice packings E_8 and Λ_{24}. Cohn and Elkies conjectured that there exist functions which optimize the Cohn-Elkies bounds in these dimensions – so-called "magic functions." These magic functions could be numerically approximated (and indeed, Cohn and Miller made several important conjectures about the nature of these functions from the obtainable numerical data [9]), but whether or not the methods employed to compute them would actually converge,[1] let alone an explicit construction of such an optimal function, remained unknown for the next decade or so.

Then, on March 14, 2016 (a Pi Day revelation), Maryna Viazovska published to the ArXiv a paper giving a construction of the magic function in dimension 8, proving at last that the E_8 lattice gives the densest sphere packing in 8 dimensions [38]. Within a fortnight, Cohn, Kumar, Miller, Radchenko, and Viazovska published a similar proof, constructing the magic function for dimension 24, showing that the Leech lattice gives the densest sphere packing in 24 dimensions [8]. (In both of these cases it is conjectured that these magic functions are unique.) Fascinatingly, the magic functions constructed by these arguments turn out to be Laplace transforms of modular (or, more precisely, quasimodular) forms. Viazovska's work establishes yet another, quite surprising, connection between sphere packings and the theory of modular forms.

These results are also notable because, unlike the proof of the Kepler conjecture, they do not require any onerous computer verification. While computers can facilitate some of the rote arithmetic involved in the arguments, they are not strictly speaking necessary; moreover, the proofs have none of the attrition that so characterized to proof of the optimal packing for $d = 3$.

1.2. **Generalities of Sphere Packings.** We consider d-dimensional Euclidean space, $\mathbb{R}^d$, equipped with the standard inner product. We let $\mathrm{Vol}(R)$ denote the volume of a region $R \subset \mathbb{R}^d$ (R measurable). Finally, we let $B_d(x, r) = \{y \in \mathbb{R}^d : \|x - y\| < r\}$ represent the open ball with center x and radius r. Recall that:

$$\mathrm{Vol}(B_d(0, r)) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)} r^d.$$

Likewise, we define the open cube $C_d(x, r) := \{y = (y_1, \ldots, y_d) \in \mathbb{R}^d : |x_i - y - i| < r\}$. This has volume $(2r)^d$.

We now take $X \subset \mathbb{R}^d$ to be a discrete set of points such that for all x and y in X, $\|x - y\| \geq 2$. These will be the centers of our spheres; they are chosen so that $B_d(x, 1)$ is disjoint from $B_d(y, 1)$ for all $x \neq y$, $x, y \in X$. We take the union:

[1]In fact, the problem of the convergence of the approximations given by Miller and Cohn remains unknown.

$$R = \bigcup_{x \in X} B_d(x, 1)$$

This region R is what we shall define to be a *sphere packing*. If X happens to be a lattice, we refer to the packing as a *lattice packing*; if X is the the union of several translates of a single lattice, we will say it is *periodic*. Otherwise, we say the packing is *irregular*.

We wish to find the *densest* packing of spheres in d-dimensional space; as we are referring to *all* of (infinite) $\mathbb{R}^d$, our packings (if they should have a nonzero density) should contain an infinite number of balls and our notion of density will implicitly depend on some sort of limit. A natural way to define this limit is to consider the proportion of space taken up by our sphere packing within some (large) cube[2] and let the side-length of the cube tend to infinity. Thus we let:

$$\Delta_R(r) := \frac{\mathrm{Vol}(R \cap C_d(0, r))}{\mathrm{Vol}(C_d(0, r))}.$$

where $C_d(x, r) = \{(y_{1,d}) \in \mathbb{R}^d : |y_i - x_i| \leq r\}$ is the cube of side-length $2r$ centered at x (note that $\mathrm{Vol}(C_d(0, r) = (2r)^d)$.

Of course, it is not hard to construct examples[3] for which $\Delta_R(r)$ fails to stabilize as $r \to \infty$; so we define the *density*[4] of a lattice packing as the limit superior of $\Delta_R(r)$:

$$\Delta_R := \limsup_{r \to \infty} \Delta_R(r).$$

Likewise, we define the *center density* of R as the dimensionless quantity:

$$N_R = \frac{\Delta_R}{\mathrm{Vol}(B_d(0, 1)} = \frac{\Delta_R \Gamma(\frac{d}{2} + 1)}{r^d \pi^{d/2}}.$$

This represents the "number of sphere centers per unit volume."

The number of interest to sphere packing theory is the limit superior of Δ_R over all sphere packings R:

$$\Delta_d := \sup_R \Delta_R$$

It is natural to ask whether, given such a Δ_d, there exists an actual sphere packing that achieves this supremal density. The answer is yes, as was shown by H. Groemer [17]. While we will not require this result, as we will only consider extremal cases coming from sphere packings that are already known, but we shall nevertheless offer a proof.

[2]One can of, course define this by taking a large *sphere* whose radius tends to infinity. This is how density is defined in [38]. We use cubes, as it is easier to prove, using this definition, that an arbitrary sphere packing can be approximated arbitrarily well by a periodic packing.

[3]Consider, for example, a lattice packing ($\mathbb{Z}^d$, say, for simplicity), in which we remove all spheres whose centers have a magnitude between $(2k)!$ and $(2k + 1)!$ for all $k \in \mathbb{Z}$.

[4]Cohn [6] calls this the *upper density*. We are following the terminology of Viazovska [38].

Let R_1, $R_2, \ldots$ be sphere packings whose densities approach a constant Δ_d, the supremum of densities over all packings. We must find a single sphere packing whose density actually equals Δ_d. To do so, we will use the "turducken method," enclosing each some finite portion of the packing R_i in a very-much-larger shell of spheres from R_{i+1}, so that, in the limit, the density of the the packing equal Δ_d.

Say that the densities of these packings are D_1, $D_2, \ldots$ where $D_i \to \Delta_d$. For each packing R_i, we can find, for any $\epsilon_i > 0$, a sequence of numbers r_{ij} where $r_{ij} \to \infty$ as $j \to \infty$ such that

$$\left| \frac{\text{Vol}(R_i \cap C_d(0, r_{ij}))}{\text{Vol}(C_d(0, r_{ij}))} - D_i \right| < \epsilon_i$$

for all j. (Moreover, D_i is by definition the supremum of all numbers with this property.)

Now, for a packing R, we can remove all spheres which intersect the interior of some finite cube $C(0, k)$ without changing the packing density. For say we removed all spheres in R that intersect the interior of $C(0, k)$. The union of all such spheres is completely contained in $C(0, k+2)$, which has volume $(2(k+2))^d$: if we call the total volume of the spheres we are removing K, we have $K < (2(k+2))^d$. In fact, we can "refill" the newly emptied chasm in $C(0, d)$ with spheres from any other packing while also not affecting the packing density; if K' is the volume of the spheres we have added in $C(0, d)$ (again, $K' < (2(k+2))^d$) then the new density of this sphere packing is given by:

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\text{Vol}(R \cap C_d(0, r)) - K + K'}{\text{Vol}(C_d(0, r))} &= \limsup_{r\to\infty} \frac{\text{Vol}(R \cap C_d(0, r))}{\text{Vol}(C_d(0, r))} + \frac{K' - K}{(2r)^d} \\
&= \limsup_{r\to\infty} \frac{\text{Vol}(R \cap C_d(0, r))}{\text{Vol}(C_d(0, r))} \\
&= \Delta_R.
\end{aligned}$$

Now, let R^* be a packing that agrees with R for all spheres that do not intersect $C(0, k_i)$, k_i a number depending on i for each i. Let D be the density of R. Then we see that:

$$\left| \frac{\text{Vol}(R_i^* \cap C_d(0, r))}{\text{Vol}(C_d(0, r))} - \frac{\text{Vol}(R_i \cap C_d(0, r))}{\text{Vol}(C_d(0, r))} \right| \leq \frac{(2(k_i + 2))^d}{(2r)^d}.$$

Given some $\epsilon_i > 0$, we can find an N so large that for all $r > N$, the LHS of this is less than $\epsilon/2$. Likewise, by the definition of the limit superior, we can find a sequence of radii $\{r_j\}$ so that $r_j \to \infty$ as $j \to \infty$, and

$$\left| \frac{\text{Vol}(R \cap C_d(0, r_j))}{\text{Vol}(C_d(0, r_j))} - D \right| < \frac{\epsilon}{2}$$

for all j. Restricting to those $r_j > N$, we see that there exists a sequence such that:

$$\begin{aligned}\left|\frac{\text{Vol}(R^* \cap C_d(0,r_j))}{\text{Vol}(C_d(0,r_j))} - D\right| &\leq \left|\frac{\text{Vol}(R^* \cap C_d(0,r_j))}{\text{Vol}(C_d(0,r_j))} - \frac{\text{Vol}(R \cap C_d(0,r_j))}{\text{Vol}(C_d(0,r_j))}\right| \\ &\quad + \left|\frac{\text{Vol}(R_i \cap C_d(0,r_j))}{\text{Vol}(C_d(0,r_j))} - D\right| \\ &\leq \epsilon,\end{aligned}$$

In other words, for a given $\epsilon > 0$ we can find a sequence of r_j, so that $r_j \to \infty$ as $j \to \infty$, such that

$$\left|\frac{\text{Vol}(R^* \cap C_d(0,r_j))}{\text{Vol}(C_d(0,r_j))} - D\right| < \epsilon$$

for all j and for all sphere packings R^* which agree with one another outside of $C(0,k)$. (The interesting content here is that we can pick the same sequence for all such R^*.)

Applying this to the sequence of sphere packings R_i of density $D_i \to \Delta_d$, we observe that we can now say, given an arbitrary sequence of numbers $k_i > 0$, and $\epsilon_i > 0$ as before, that there exists a series of r_{ij}, depending on k_i and ϵ_i, with $r_{ij} \to \infty$ as $j \to \infty$ for each i, such that, for all j,

$$\left|\frac{\text{Vol}(R_i^* \cap C_d(0,r_{ij}))}{\text{Vol}(C_d(0,r_{ij}))} - D_i\right| < \epsilon_i$$

for *any* sphere packing R_i^* that agrees with R_i for all spheres that do not intersect the interior of $C_d(0,k_i)$.

We will now use this fact to construct a single sphere packing R of density Δ_d. First, we let $\epsilon_i = 1/i$. Then, we define $R_1' = R_1$. Now, let $k_1 = 0$. Note that this and the fact that $\epsilon_1 = 1$ gives rise to a sequence $\{r_{1j}\}$ with the above-defined properties.

Now, inductively assume that we have defined R_i' and k_i for a given i (we have already specified $\epsilon_i = 1/i$ for all i). Then there exists a sequence $r_{i,j}$ for all j such that, as discussed above, $r_{ij} \to \infty$ as $j \to \infty$, and

$$\left|\frac{\text{Vol}(R_i^* \cap C_d(0,r_{ij}))}{\text{Vol}(C_d(0,r_{ij}))} - D_i\right| < \epsilon_i$$

for all j. Because $r_{ij} \to \infty$ as $j \to \infty$, we can pick an r_{ij} arbitrarily large; say, so that $r_{ij} > \max(i, k_i)$. Let us call such a number $r^{(i)}$. We define $k_{i+1} = r^{(i)}$. Then we can define R_{i+1}' to equal R_i' for all spheres whose interiors lie inside $C_d(0,k_{i+1})$ and to equal R_i for all spheres whose interiors lie in the exterior of $C_d(0,k_{i+1})$.

This produces a series of sphere packings R_i' that agree with one another for those spheres whose interiors are contained with in the cube $C_d(0,k_n)$ for all $i > n$. Moreover, we have rigged k_n to increase and go to infinity. Thus we can speak of the stable limit of this sphere packing; let us call it R. Note that, by construction:

$$\left|\frac{\mathrm{Vol}(R \cap C_d(0,k_i))}{\mathrm{Vol}(C_d(0,k_i))} - D_i\right| < 1/i.$$

Moreover, we know, that $D_i \to \Delta_d$ as $i \to \infty$, so that

$$\frac{\mathrm{Vol}(R \cap C_d(0,k_i))}{\mathrm{Vol}(C_d(0,k_i))} \to \Delta_d$$

as $k_i \to \infty$. Thus, we know, by the definition of limit superior, that the packing density of Δ_R is at least Δ_d; however, Δ_d has been defined to be the supremal density among all sphere packings, so we have $\Delta_R = \Delta$.

Note that R is very much irregular (even if the R_i are regular); it is unknown if there exists a supremal sphere packing which is periodic. However, the following is true:

Lemma 1.2.1 Any sphere packing can be arbitrarily well approximated by periodic sphere packings.

This will be the basic sphere packing fact used by the Cohn-Elkies linear programming bound, so we will give a complete proof presently.

Proof. Say that we have a sphere packing R, of density Δ_R. Given an $\epsilon > 0$, we will find a periodic packing R' so that $|\Delta_R' - \Delta_R| < \epsilon$.

Given such an ϵ, we can find arbitrarily large radii r such that

$$\left|\frac{\mathrm{Vol}(R \cap C_d(0,r))}{\mathrm{Vol}(C_d(0,r))} - \Delta_R\right| < \epsilon/2. \tag{1}$$

Note that the set of points contained by those spheres of R that intersect the boundary of the cube $C_d(0,r)$ is properly contained by the cubic shell of distance two on either side of each face; this volume is in turn less than $(2d)(2r)^{d_1} \cdot 4$ – coming from $2d$ faces, each of area $(2r)^{d-1}$, with a possible bleeding of 2 units in either direction.

We now choose r so big that (1) is satisfied and so that $4d/r = ((2d)(2r)^{d-1}\cdot 4)/((2r)^d) < \epsilon/2$. We can do so as the collection of r satisfying (1) is unbounded. Now, remove from R all cubes whose interiors intersect the boundary of cube $C_d(0,r)$. Call the collection of remaining spheres R'. We have:

$$\left|\frac{\mathrm{Vol}(R' \cap C_d(0,r))}{\mathrm{Vol}(C_d(0,r))} - \frac{\mathrm{Vol}(R \cap C_d(0,r))}{\mathrm{Vol}(C_d(0,r))}\right| \le \frac{(2d)(2r)^{d-1}\cdot 4}{(2r)^d} < \epsilon/2.$$

Thus, by the triangle inequality, we have:

$$\left|\frac{\mathrm{Vol}(R' \cap C_d(0,r))}{\mathrm{Vol}(C_d(0,r))} - \Delta_R\right| < \epsilon. \tag{2}$$

Now, we proceed to translate the collection of spheres S by the cubic lattice $(2r)\cdot\mathbb{Z}^n \subset \mathbb{R}^n$. By construction this lattice is periodic, and has density exactly equal to

$$\frac{\mathrm{Vol}(R' \cap C_d(0,r))}{\mathrm{Vol}(C_d(0,r))}.$$

Thus R' is a periodic packing, and by (2), has a density within ϵ of Δ_R.

1.3. E_8 **and the Leech Lattice.** As we shall see, the optimal sphere packings in dimension 8 and 24 are two famous lattice packings: E_8 and the Leech lattice, Λ_{24}. For various constructions, as well as a detailed discussion, of these extraordinary lattices and their remarkable properties, we refer the reader to the compendious *Sphere Packings, Lattices and Groups* of Conway and Sloane (often referred to as "SPLAG") [10].

Rather fascinatingly, Viazovska's argument and the corresponding argument in dimension 24 does not make much use of the exceptional properties of these lattices. As a result, we have not decided to give an exposition of them, and refer the reader to the above reference for more details. In fact, all we will need are Theorems 1.3.1 and 1.3.2, whose proofs are also to be found in [10]. For a reader not willing to take these results on faith, the rest of this document will work towards proving that, for a lattice satisfying the hypotheses of Theorems 1.3.1. and 1.3.2., the associated sphere packing is the densest possible.

We introduce the following terminology. A *lattice* is a free, rank n subgroup of $\mathbb{R}^n$ with the usual inner product. A lattice is said to be *integral* is $\langle x, y\rangle \in \mathbb{Z}$ for all x and y in Λ. A lattice is said to be *even* if the square magnitudes of the lattice vectors are all even integers. For every lattice Λ, there exists a *dual lattice*, Λ^*, defined as the set of all vectors x such that for all $y \in \Lambda$, $\langle x, y\rangle \in \mathbb{Z}$. If $\Lambda_* = \Lambda$, then Λ is said to be *self-dual*. Given a basis $e_1, \ldots, e_n$ for a lattice, we can take construct an $n \times n$ matrix whose (i, j)th component is $\langle e_i, e_j\rangle$. This is called a *Gram matrix*, and the determinant of the Gram matrix is called the *determinant* of the lattice. If the the lattice has determinant is ± 1, it is said to unimodular. An integral unimodular lattice is self-dual. Two lattices are isomorphic if there is a bjection between the two lattice vectors coming from an orthogonal linear map $\mathbb{R}^n \to \mathbb{R}^n$.

Theorem 1.3.1. There exists a lattice in $\mathbb{R}^8$ that is even and unimodular. This lattice is unique up to isomorphism.

The unique lattice described by this theorem is E_8. Explicitly, we can construct E_8 as:

$$\left\{ x = (x_1, \ldots, x_8) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 : \sum_{i=1}^{8} x_i \equiv 0 \mod 2 \right\}$$

In other words we find the subset of the integral lattice $\mathbb{Z}^8$ and the translated "affine" lattice constructed by adding to every lattice element in $\mathbb{Z}^8$ the vector $(1/2, \ldots, 1/2)$. Then we take the subset of those vectors such that the sum of the components of each vector is an even integer. This gives one construction of E_8, and from this construction, one can easily compute that E_8 is even and unimodular. The uniqueness of a lattice with such properties, however, is more subtle; several proofs exist (see [10; 13; 26]).

Theorem 1.3.2. In $\mathbb{R}^{24}$, there exists a unique even, unimodular lattice, none of whose vectors have magnitude $\sqrt{2}$.

For a proof of this, see chapter 12 of [10] (the result is due to Conway). This lattice is called the Leech lattice. There are myriad constructions for this lattice; over 20 are listed in [10]. We shall give the following construction, which points out a rather fascinating connection between the Leech lattice and the Diophantine equation $1^2 \ldots + k^2 = N^2$, whose only solution after $k = 1$ and $N = 1$ is $k = 24$ and $N = 70$.[5]

We define on $\mathbb{R}^{26}$ the Lorentzian inner product: if $x = (x_1 \ldots x_{26}) \in \mathbb{R}^{26}$ and $y = (y_1 \ldots y_{26}) \in \mathbb{R}^{26}$, we let $\langle x, y\rangle_L := x_1y_1 + \ldots + x_{25}y_{25} - x_{26}y_{26}$. We define the lattice[6] $\mathrm{II}_{25,1}$ as:

$$\left\{ x = (x_1, \ldots, x_{26}) \in \mathbb{Z}^{26} \cup \left(\mathbb{Z} + \frac{1}{2}\right)^{26} : \sum_{i=1}^{26} x_i \equiv 0 \mod 2 \right\}$$

Now, within $\mathrm{II}_{25,1}$ there exists a special vector:

$$w := (1, 2, \ldots, 24, 70)$$

which has magnitude 0 and integer components. We see, therefore, that $w \in w^\perp$. We take $w^\perp/w$, giving a 24-dimensional lattice; this is the esteemed Leech lattice Λ_{24}.

2. The Elementary Theory of Modular Forms

We will now give an introduction to the theory of modular forms. This exposition follows [41] very closely – we follow the structure of Zagier's exposition, choosing to omit those parts not relevant to our later work on sphere packing.

2.1. First definitions.

Let $\mathfrak{H}$ denote the upper half plane; that is:

$$\mathfrak{H} = \{z = a + bi \in \mathbb{C} : b > 0\}.$$

The group $\mathrm{SL}(2, \mathbb{R})$, of 2×2 determinant 1 matrices with real entries, acts on $\mathfrak{H}$ by fractional linear transformations (also called Möbius trnasformations):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) := \frac{az + b}{cz + d}$$

for $z \in \mathfrak{H}$.

[5]This is often posed as the so-called "cannonball problem": one has pyramid of cannonballs, given by one cannonball on top, a square of four cannonballs underneath, and so-on until one has a $k \times k$ square of cannonballs at the bottom. How many cannonballs should one have, and what is the size of the bottom row, so that the total number of cannonballs is a perfect square? We thought it fitting to mention this presentation of the problem given its sphere packing resonances. The first mathematician to conjecture that these are the only solutions to this particular Diophantine equation was Eduard Lucas [24]; it was proved by G.N. Watson in 1918 [39]. More recent, elementary proofs have been offered [1; 25].

[6]Here we are appealing a slightly more general definition of a lattice. Recall that we defined a lattice as a free Abelian group of rank n embedded into $\mathbb{R}^n$ endowed with the standard inner product. Here we define a lattice as a a free Abelian group of rank 26 embedded into $\mathbb{R}^{26}$ endowed with the Lorentzian inner product. This inner product is not merely non-standard; it is not positive definite.

It is a straightforward computation to verify that composing fractional linear transformations is equivalent to multiplying the corresponding matrices; a more revealing reason for this behavior is given by viewing Möbius transformations as linear functions on complex projective space: $\mathbb{P}^1(\mathbb{C}) = \mathbb{P}(\mathbb{C}^2)$. A matrix acts linearly $\mathbb{C}^2$ and so projects to an aciton on $\mathbb{P}^1$: in homogeneous coordinates on $\mathbb{P}^1$, a linear function sends $[x, y] \mapsto [ax + by, cx + dy]$. In inhomogeneous coordinates, $[x, 1]$, we have $[x, 1] \mapsto \left[\frac{ax+b}{cx+d}, 1\right]$, which shows how Möbius transformations act like 2×2 matrices (and so must compose like 2×2 matrices).

To see that Möbius transformations actually map $\mathfrak{H}$ to $\mathfrak{H}$, we let $z = u + vi$; $u, v \in \mathbb{R}$ and $v > 0$. We let $\mathrm{Im} z$ denote the imaginary part of z; in paticular, $\mathrm{Im}(z) = v$. We compute:

$$\begin{aligned} \mathrm{Im}\left(\frac{az+b}{cz+d}\right) &= \mathrm{Im}\left(\frac{au+avi+b}{cu+cvi+d}\right) \\ &= \mathrm{Im}\left(\frac{au+b+avi}{cu+d+cvi} \cdot \frac{cu+d-cvi}{cu+d-cvi}\right) \\ &= \frac{(ad-bc)v}{|cz+d|^2} \\ &= \frac{v}{|cz+d|^2} \qquad (3) \end{aligned}$$

In particular, the imaginary part of $\frac{az+b}{cz+d}$ has the same same sign as that of z, proving that $\gamma : \mathfrak{H} \to \mathfrak{H}$ for all $\gamma \in \mathrm{SL}(2, \mathbb{R})$.

Note that the matrices M and $-M$ represent the same fractional linear transformation; this suggests that most natural group to to consider is not $\mathrm{SL}(2, \mathbb{R})$ but $\mathrm{PSL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R})/\{I, \pm I\}$. (The "modular group," $\mathrm{PSL}(2, \mathbb{Z})$, has a trivial stabilizer in acting on $\mathfrak{H}$.)

Modular forms are holomorphic functions on $\mathfrak{H}$ which transform in a prescribed manner under the action of a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$. For our purposes, we will consider only the class of "congruence subgroups" of $\mathrm{SL}(2, \mathbb{Z}) := \Gamma(1)$. The most basic congruence subgroups of $\Gamma(1)$ are often written $\Gamma(N)$, $\Gamma_1(N)$, and $\Gamma_0(N)$. They are defined as follows:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$

where $*$ represents an arbitrary integer. A subgroup $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$ is called a *congruence subgroup* if it contains $\Gamma(N)$ for some N; the minimal choice of N such that $\Gamma \supseteq \Gamma(N)$ is called the *level* of the subgroup.

Given any Γ, a discrete subgroup of $\mathrm{SL}(2,\mathbb{Z})$, [7] we define a *modular form of weight k on* Γ to be a function, holomorphic on $\mathfrak{H}$, satisfying:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \tag{4}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and of "subexponential" growth at infinity; that is:

$$f(x+iy) = O\left(e^{Cy}\right) \text{ as } y \to \infty \text{ for all } C > 0 \tag{5}$$

and

$$f(x+iy) = O\left(e^{C/y}\right) \text{ as } y \to 0 \text{ for all } C > 0. \tag{6}$$

(These analytical conditions will be explained shortly.)

An important coherence condition implied by the transformation property must be observed. Namely, we must have

$$f((\gamma\gamma')(z)) = f(\gamma(\gamma'(z))$$

for all $\gamma, \gamma' \in \Gamma$. Letting

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

we see, by applying transformation formula (4) to the left hand and right-hand side, that

$$((ca' + dc')z + (cb' + dd'))^k f(z) = \left(c(c'z + d') + d\right)^k (c'z + d')^k f(z).$$

A quick computation reveals that this identity is satisfied identically – i.e., it "comes for free." It expresses an important property of the so-called "factor of automorphy" $(cz + d)^k$; namely, if we let $F(\gamma, z) = (cz + d)^k$, then

$$F(\gamma\gamma', z) = F(\gamma, \gamma' z)F(\gamma', z). \tag{7}$$

This is called the *cocycle condition*, as a function satisfying (7) called a 1-cocyle in group cohomology.

[7]This definition applies to congruence as well as non-congruence subgroups, although we we will only consider congruence subgroups here.

The set of modular forms of a specified weight on Γ constitutes a complex vector space, which we denote $M_k(\Gamma)$. The space

$$\bigoplus_k M_k(\Gamma) := M_*(\Gamma)$$

forms a graded algebra over $\mathbb{C}$ under the usual product of functions: the product of a modular form of weight k and a modular form of weight l is a modular form of weight $k+l$. As we shall prove, each of the spaces $M_k(\Gamma)$ is finite-dimensional, and the algebra $M_*(\Gamma)$ will turn out to be finitely generated over $\mathbb{C}$.

Let us consider the matrices of the form

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},\ k \in \mathbb{Z}$$

in some Γ. These correspond to the transformations of the form $z \mapsto z+k$; i.e., horizontal shifts. The transformation formula (4) gives

$$f(z+k) = f(z). \tag{8}$$

The minimal k such that (8) holds for all z is called the period of f. The existence of such a period implies that we can develop f in a Fourier series. Letting $q = e^{2\pi i z}$, we can write:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^{n/k}$$

The growth conditions (5) and (6) imply that there are no negative-index Fourier coefficients. For let $A(z) = \sum_n a_n z^n$; i.e., $A\left(e^{2\pi i z/k}\right) = f(z)$. By Cauchy's integral formula:

$$a_n = \frac{1}{2\pi i} \oint \frac{A(z)}{z^{n+1}} dz.$$

Considering this contour integral about a circle of radius ϵ with $0 < \epsilon < 1$, we get:

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \oint \frac{A\left(\epsilon e^{2\pi i t}\right)}{\left(\epsilon e^{2\pi i t}\right)^{n+1}} \cdot \epsilon e^{2\pi i t}(2\pi i) dt \right| \\ &\leq \frac{1}{\epsilon^n} \int_0^1 A\left(e^{2\pi i t + \log(\epsilon)}\right) dt \\ &= \frac{1}{\epsilon^n} \int_0^1 A\left(e^{2\pi i (t - [i \log(\epsilon))/2\pi])}\right) dt \\ &= \frac{1}{\epsilon^n} \int_0^1 f\left(k\left[t - \frac{i \log(\epsilon)}{2\pi}\right]\right) dt. \end{aligned}$$

Noting that $0 < \epsilon < 1$, we see that $\log(\epsilon) < 0$, so that as $\epsilon \to 0$, the imaginary part of f's input tends to $+\infty$. Thus we can use our growth condition, and get the bound:

$$\begin{aligned} |a_n| &\leq \frac{1}{\epsilon^n} e^{-K\log(\epsilon)} \\ &= \epsilon^{-(n+K)} \end{aligned}$$

for all $K > 0$. If $n \leq -1$, then we can let $K < 1$, and, as $\epsilon \to 0$, the bound on the RHS tends to 0 as well. Thus, for $n < 0$, our Fourier coefficient a_n vanishes of necessity.

As discussed in Zagier [41], much of the great utility of modular forms stems from this Fourier expansion in combination with the finite-dimensionality of $M_k(\Gamma)$ for each k. Indeed – and this is one of the most tantalizing facts in mathematics – the Fourier coefficients, a_n, of modular forms tend to be of profound interest in other areas of mathematics; given this finite dimensionality, we can thus often find unexpected linear combinations among these sequences. Moreover, to prove that a given such relation holds in general, we need only check that it holds for a finite number of terms.

2.2. **The Fundamental Domain of the Modular Group.** If we know how a function f transforms under certain symmetries, it often suffices to define f in some restricted domain and then to use the symmetry to define it everywhere else. To perform this process, we must make sure that the translates of the restricted domain, under the action of the group, do not overlap (or overlap as little as possible). This way we can choose our original f on the restricted domain as freely as possible.

Let us examine an illustrative example. Say we want to construct an even function; that is, $f : \mathbb{R} \to \mathbb{R}$ such that $f(-x) = f(x)$. We need only define f on all $x > 0$; the rest is then specified by the symmetry constraints. If we insist that f be an *odd* function – that is, $f(-x) = -f(x)$ – then the domain is the same, although now we must also insist that $f(0) = 0$. As a final example, consider the case of periodic functions; say $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x)$. In this case we need only define f on the unit interval $[0, 1]$; although, like with the case of odd functions, we must be careful that the boundaries agree: i.e., that $f(0) = f(1)$.

We are getting at the idea of a "fundamental domain." If a group Γ acts on a topological space X via continuous transformations, a *fundamental domain* for Γ is an open set U such that no two distinct points of U are in the same Γ-orbit, while every point of X lies in the orbit of some point in the closure, $\overline{U}$.

Theorem. A fundamental domain of $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$ (or equivalently, the modular group $\mathrm{PSL}(2, \mathbb{Z})$), acting on $\mathfrak{H}$ by fractional linear transformations, is given by:[8]

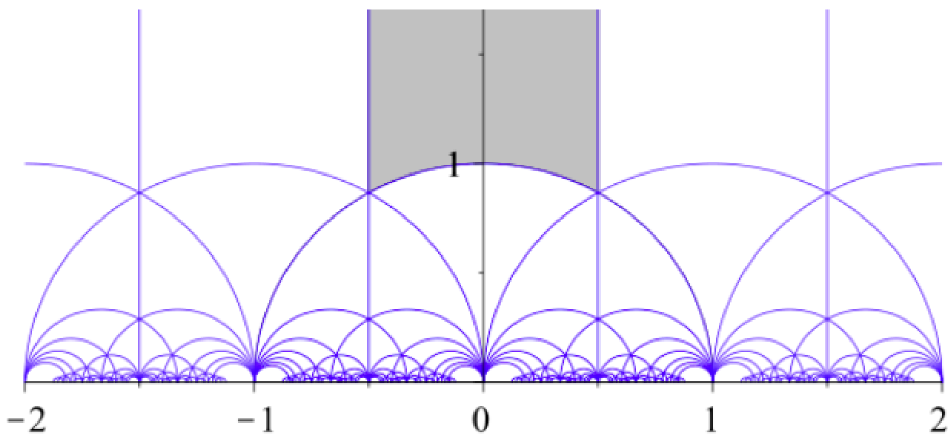
$$\mathcal{F} = \left\{ z \in \mathfrak{H} \ : \ |z| > 1, |\mathrm{Re}(z)| < \frac{1}{2} \right\}.$$

[8]This shape is a hyperbolic triangle: the boundaries o $\mathcal{F}$ are geodesics of $\mathbb{H}$ under the Poincare Metric. PSL(2,R) acts on $\mathbb{H}$ via isometries, so geodesics are mapped to geodesics, and thus the tiling of $\mathbb{H}$ by $\Gamma(1)$-translates of F defines a tessellation of $\mathbb{H}$ via hyperbolic triangles.

(Here $\mathrm{Re}(z)$ refers to the real part of z.)

This fundamental domain is depicted below, in grey, along with the so-called *upper half plane tiling* of $\mathfrak{H}$ by the $\Gamma(1)$-translates of $\mathcal{F}$ (the blue lines are the boundaries, and the vertical black line is the imaginary axis).

FIGURE 1. The Upper Half Plane tiling.



Proof. Let $z \in \mathfrak{H}$. Consider the complex lattice $\{mz + n : m, n \in \mathbb{Z}\}$. There must exist a nonzero value in this lattice of minimal complex magnitude, as the lattice is a discrete subset of $\mathbb{C}$ and 0 lies in the lattice. Let us call element $cz + d$. If c and d shared a common factor $k > 1$, we could divide it out, and get a new lattice element $(c/k)z + (d/k)$ of strictly smaller magnitude. Thus c and d are relatively prime; hence, by Bezout's lemma, there exist integers a and b such that $ad - bc = 1$. Equivalently, we can find a and b such that

$$\alpha := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1).$$

Recalling (3), we see that $\mathrm{Im}(\alpha(z))$ is as large as possible for all $\mathrm{Im}(\gamma(z))$, $\gamma \in \Gamma$. We now repeatedly apply

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{9}$$

to shift $\alpha(z)$ by unit increments until $z^* = T^n(\alpha(z))$ satisfies $|\mathrm{Re}(z^*)| \leq \frac{1}{2}$.

Now $\mathrm{Im}(z^*) = \mathrm{Im}(\alpha(z))$, as we have only translated $\alpha(z)$ by (real) unit increments. Thus, $\mathrm{Im}(z^*)$ is also as large as possible. Thus $|z^*| \geq 1$ else we could apply

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

(which corresponds to the transformation $z \mapsto -\frac{1}{z}$) and get $\text{Im}(Sz^*) = \frac{\text{Im}(z^*)}{|z^*|^2} > \text{Im}(z^*)$, contradicting maximality. Thus we see that $z^* = T^n(\alpha(z))$ is in the $\Gamma(1)$-orbit of z, and satisfies $|\text{Im}(z^*)| \leq \frac{1}{2}$, and $|z^*| > 1$. In other words, $z^* \in \overline{\mathcal{F}}$.

We next check that no two points within $\mathcal{F}$ are mapped to one another by an element of $\Gamma(1)$. Say to the contrary that z_1 and z_2 are both in the same Γ-orbit; that is, $z_2 = \gamma(z_1)$, where $\gamma \neq \{I, -I\}$. Then $\gamma \neq T^n$, as $|\text{Re}(z_1)|$ and $|\text{Re}(z_2)|$ are both strictly less than 1/2. Thus $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c \neq 0$. (Observe that if $c = 0$, the integers along the main diagonal must both be $+1$ or -1, as $\det(\gamma) = 1$. We may assume that both are $+1$ as $\gamma \sim -\gamma$ in $\text{PSL}(2, \mathbb{Z})$). Now, observe that $\text{Im}(z_i) > \frac{\sqrt{3}}{2}$. We have:

$$\frac{\sqrt{3}}{2} < \text{Im}(z_2) = \text{Im}(\gamma z_1) = \frac{\text{Im}(z_1)}{|cz_1 + d|^2}$$

We can give a lower bound to the denominator of this last term by examining only the imaginary-part component:

$$\frac{\text{Im}(z_1)}{|cz_1 + d|^2} \leq \frac{\text{Im}(z_1)}{c^2 \text{Im}(z_1)^2} = \frac{1}{c^2 \text{Im}(z_1)} < \frac{2}{\sqrt{3}c^2}.$$

Together, these imply that $\frac{\sqrt{3}}{2} < \frac{2}{\sqrt{3}c^2}$, which means that $c = \pm 1$. We can assume, without loss of generality, that $\text{Im}(z_1) \leq \text{Im}(z_2)$ (otherwise, reverse the roles of z_1 and z_2 and use γ^{-1} instead of γ). If $c = \pm 1$, then:

$$\text{Im}(z_2) = \text{Im}(\gamma z_1) = \frac{\text{Im}(z_1)}{|\pm z_1 + d|^2} \leq \frac{\text{Im}(z_1)}{|z_1|^2} < \text{Im}(z_1)$$

as $|\pm z_1 + d| \geq |z_1| > 1$ (the first inequality following from $|\text{Re}(z)| < \frac{1}{2}$). This is a contradiction, and the theorem follows.

□

An immediate consequence of this theorem is that $\text{PSL}(2, \mathbb{Z})$ is generated by the above-defined matrices S and T.[9] For, if we examine the above image of the upper-half-plane tiling, we will see that the neighboring hyperbolic triangles are $T\mathcal{F}$, $T^{-1}F$ (to the left and right, respectively), and $S\mathcal{F}$ (below). Given some translate $\gamma\mathcal{F}$, we can transform it to one of its three neighbors via applying $\gamma T \gamma^{-1}$, $\gamma T^{-1} \gamma^{-1}$, or $\gamma S \gamma^{-1}$. In particular, we can apply the transformation S and T repeatedly to $\mathcal{F}$ to bring $\mathcal{F}$ to its neighbors, its neighbors' neighbors, etc. and so eventually (arguing by induction) to any $\gamma\mathcal{F}$.

Thus both γ and some word in S and T bring $\mathcal{F}$ to $\gamma\mathcal{F}$. We showed above that no two points in $\mathcal{F}$ lie in the same $\text{PSL}(2, \mathbb{Z})$-orbit, so, clearly, only the trivial element of $\text{PSL}(2, \mathbb{Z})$ stabilizes the whole tile $\mathcal{F}$. Then, given any other fractional linear transformation $\gamma' \in \text{PSL}(2, \mathbb{Z})$ such that $\gamma' : \mathcal{F} \to \gamma\mathcal{F}$, we see that $\gamma^{-1}\gamma' : \mathcal{F} \to \mathcal{F}$, which shows that

[9] We are being somewhat glib here; elements of $\text{PSL}(2, \mathbb{Z})$ are not matrices but *pairs* of matrices $\{M, -M\}$. Of course, we mean that the pairs of matrices $\{T, -T\}$ and $\{S, -S\}$ generate $\text{PSL}(2, \mathbb{Z})$. We will often abuse terminology in this way when our meaning is clear.

$\gamma^{-1}\gamma' = e$, or $\gamma = \gamma'$. Thus we can conclude that γ equals the word in S and T, showing that S and T generate the full modular group.[10]

Thus, to check that the transformation property (2) is satisfied for a function f which we wish to show is modular on $\Gamma(1)$, the full modular group, we need only show

$$f(z+1) = f(z), \tag{11}$$

$$f(-1/z) = z^k f(z), \tag{12}$$

which together will imply (4) for all transformations in $\mathrm{PSL}(2,\mathbb{Z})$.[11]

2.3. **The Valence Formula and Finite Dimensionality of $M_k(\Gamma(1))$.** We now prove the finite-dimensionality of $M_k(\Gamma)$ for Γ the full modular group. We will do so by first proving an elegant property concerning the integral of the logarithmic derivative of a modular form of weight k along the boundary of our fundamental region $\mathcal{F}$, sometimes called the valence formula. This will supply an upper bound for the dimension of $M_k(\Gamma(1))$.

We first will deal quite generally with a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$. We will study the geometry of $\mathbb{H}/\Gamma$.

If $f(z) = 0$ for some $z \in \mathbb{H}$, then $f\left((az+b)/(cz+d)\right) = (cz+d)^k f(z)$ also vanishes; moreover, it vanishes to the same order. Thus we can associate to each point $P \in \mathbb{H}/\Gamma$ a well-defined number, which we will call the "order" of f at P, given by the order of vanishing of f at any points in $\mathbb{H}$ which cover P.

Some points of $P \in \mathbb{H}/\Gamma$ will be "singular" – that is, we can lift P to some $\widetilde{P} \in \mathbb{H}$ such that $\widetilde{P}$ will be stabilized by a nontrivial subgroup of Γ. By the orbit-stabilizer theorem, the subgroups of Γ which stabilize *any* choice of lift $\widetilde{P}$ will be isomorphic as groups (in fact, conjugate as subgroups of Γ). We let n_P denote the order of this stabilizer, which is independent of choice of lift $\widetilde{P}$. In fact, the stabilizer of any point in $\mathbb{H}$ under the action of Γ will be a cyclic group of order n_P: the $\mathrm{SL}(2,\mathbb{R})$-stabilizer of a point in $\mathbb{H}$ is simply the group $S^1 \times \{I, -I\}$, a union of two disjoint circles. The intersection of this subgroup with a discrete group $\Gamma \subset \mathrm{SL}(2,\mathbb{R}))$ will necessarily be cyclic, and we have already defined its order to be n_P.

From a Riemann-surface point of view, a point P whose preimage in $\mathbb{H}$ has a cyclic-group stabilizer of order $n_P > 1$ will correspond to a point of $\mathbb{H}/\Gamma$ which cannot be coordinatized by lifting some neighborhood $U \subset \mathbb{H}/\Gamma$ of $P \in \mathbb{H}/\Gamma$ to a neighborhood $\widetilde{U} \subset \mathbb{H}$ of some $\widetilde{P} \in \mathbb{H}$. (At these points, the projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ fails to be a covering map.)

For say $\pi(\widetilde{P}) = P$, and that $\widetilde{P} \in \mathbb{H}$ is a point with a nontrivial Γ-stabilizer. Let $\widetilde{U} \subset \mathbb{H}$ be any neighborhood of $\widetilde{P}$. Now, say $\gamma \in \mathrm{Stab}_\Gamma(\widetilde{P})$, with $\gamma \neq \mathrm{Id}$. The transformation γ acts continuously on $\mathbb{H}$, whence $\gamma^{-1}(\widetilde{U})$ is open in $\mathbb{H}$. Moreover, as $\gamma(\widetilde{P}) = \widetilde{P}$, we have $\widetilde{P} \in \gamma^{-1}(\widetilde{U})$. Thus $\gamma^{-1}(\widetilde{U}) \cap \widetilde{U}$ is open and nonempty. Hence there exists a point $x \in \gamma^{-1}(\widetilde{U}) \cap \widetilde{U}$

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[10] Expressing a general element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a product of the elements S and T is closely related to the expression of a/c as a continued fraction (See [3]).

[11] This might not be immediately apparent. In fact it follows from the cocycle condition, (7).

such that $x \neq P$, and we can assume that $\gamma(x) \neq x$. (Indeed, if this were impossible, then for all x in the nonempty open set $\gamma^{-1}(\widetilde{U}) \cap \widetilde{U}$, $\gamma(x) = x$. But γ is holomorphic, hence if $\gamma(x) = x$ on some open neighborhood then we would have $\gamma(x) = x$ for all $x \in \mathbb{H}$; yet we have assumed already $\gamma \neq \text{Id}$.) But then $x \neq \gamma(x)$ yet both are in $\widetilde{U}$ *and* in the same Γ-orbit. Thus $\pi(\gamma(x)) = \pi(x)$, and so π cannot map $\widetilde{U}$ bijectively onto its image $U := \pi(\widetilde{U})$.

Interestingly, however, it will still be possible to attribute a Riemann surface structure to $\mathbb{H}/\Gamma$. Indeed, around some singular point P, we must first lift to a neighborhood $\widetilde{U} \subset \mathbb{H}$ (as discussed above, $\pi : \widetilde{U} \to U$ will not be a bijection), and then post-compose by the map $(z - \widetilde{P})^{n_P}$. This (after possibly restricting U) gives a complex coordinate patch in bijection with U. In fact this procedure defines a coordinate chart for all $P \in \mathbb{H}/\Gamma$ (letting $n_P = 1$ trivially for all non-singular points $P \in \mathbb{H}/\Gamma$), and it can be easily verified that the "transition maps" between these charts are holomorphic. This gives $\mathbb{H}/\Gamma$ the structure of a Riemann surface.

Let us now determine the singular points of the quotient of the action of $\mathrm{PSL}(2, \mathbb{Z})$, the full modular group, on $\mathbb{H}$. Any point in the interior of the fundamental domain $\mathcal{F}$ gets mapped to the interior of another tile for each $\gamma \in \Gamma$; this tile-to-group-element correspondence is a bijection by the definition of a fundamental domain. Thus any singular points of $\mathbb{H}/\Gamma$ must lift to a point on the boundary of $\mathcal{F}$.

By definition, if a transformation $\gamma \in \Gamma(1)$ stabilizes a point, then $\gamma(z) = z$ for some $z \in \mathbb{H}$, and so $z \in \mathcal{F} \cap \gamma\mathcal{F}$. In particular, the only γ which can might have a fixed point is the (finite) collection of γ such that $\overline{\mathcal{F}} \cap \gamma\overline{\mathcal{F}} \neq \emptyset$.

Explicitly, these γ are I, T^{-1}, $T^{-1}S$, STS, ST, S, ST^{-1}, $ST^{-1}S$, TS, and T, where $T : z \mapsto z + 1$ and $S : z \mapsto -1/z$ as in (9) and (10). The only functions here with fixed points are Id, which fixes everything; S, which fixes i; $T^{-1}S$ and ST, which fix $\omega = \frac{-1+\sqrt{-3}}{2}$; and ST^{-1} and TS, which fix $\omega + 1 = \frac{1+\sqrt{-3}}{2}$. We see that ω and $\omega + 1$ are Γ-equivalent (the element T maps one to the other), so they correspond to the the same point in $\mathbb{H}/\Gamma$, which we shall rather abusively call ω. We see that $n_\omega = 3$ in this case, as can be seen by lifting to either to ω or to $1 + \omega \in \mathbb{H}$ – these have stabilizers $\{\text{Id}, T^{-1}S, ST\}$ and $\{\text{Id}, T^{-1}S, ST\}$, respectively. Likewise, i is stabilized by the cyclic group of order 2, given by $\{\text{Id}, S\}$. Using the same notational abuse, we say that $n_i = 2$. These are the only two singular points of $\mathbb{H}/\Gamma$ – often called the "elliptic fixed points."

We now compactify $\mathbb{H}/\Gamma$ by adding a point at infinity – a so-called "cusp." This corresponds to a single point, which we can think of as the limit of $x + iy$ as $y \to \infty$ in $\mathcal{F}$ (independent of x). Observe that the set of points in $\mathcal{F}$ with imaginary part larger than some constant $Y \geq 1$ corresponds to the values of $q = e^{2\pi iz}$ whose magnitude satisfies $0 < q < e^{-2\pi Y}$. We thus coordinatize ∞ by $q = 0$, where we take q as a local coordinate for $\mathbb{H}/\Gamma$ in the following sense: if U is some open set in $\mathbb{H}$ such that $\mathrm{Im}(z) > 1$ for all $z \in$ U, the open neighborhood in $\mathbb{H}/\Gamma$ corresponding to $\overline{U} \subset \mathbb{H}/\Gamma$ will be coordinatized by $q = e^{2\pi i\widetilde{z}}$, with $\widetilde{z}$ a lift (to $\mathbb{H}$) of a point in $\mathbb{H}/\Gamma$. (Note that this function is well-defined on the quotient, as $e^{2\pi iz}$ is periodic, and U only intersects the top row of tiles in the upper half-plane tiling.) Then we coordinatize ∞ in this local coordinate chart by $q = 0$. By introducing the point ∞ with this coordinatization, we get a compactified space $\overline{\mathbb{H}/\Gamma}$ homeomorphic to the sphere S^2. Note that ∞ is is singular – the stabilizer of ∞ is the infinite cyclic group generated by

T. (While the points of $\mathbb{H}/\Gamma$ have finite Γ-stabilizers, there is not only no obligation, but no possibility that the points of $\overline{\mathbb{H}/\Gamma}$ also have finite Γ-stabilizers.)

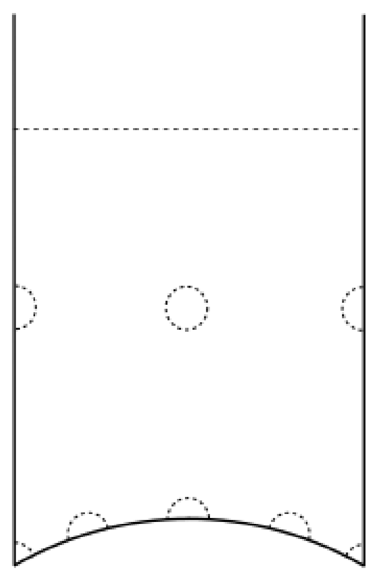
Given f, a modular form, we can see from the vanishing of the negative exponents of f's Fourier expansion that f is well-defined at ∞ – explicitly, $f(\infty) = a_0$, the constant term of the Fourier Expansion. We define $\text{ord}_\infty(f)$ to be the smallest n such that a_n is nonzero in f's Fourier expansion $\sum_{n=0}^{\infty} a_n q^n$. We can now state the following:

Theorem 2.3.1. (Valence Formula). Let f be a modular form of weight k on $\Gamma(1)$. Then:

$$\sum_{P \in \mathbb{H}/\Gamma(1)} \frac{1}{n_P} \text{ord}_P(f) + \text{ord}_\infty(f) = \frac{k}{12}.$$

Proof. We consider the region, D, defined by deleting from $\overline{\mathcal{F}}$ the intersection of $\overline{\mathcal{F}}$ with small circular neighborhoods in $\mathbb{C}$ of radius ϵ around each zero of f along with deleting the intersection of $\overline{\mathcal{F}}$ with "the neighborhood of infinity" $\text{Im}(z) > Y := \epsilon^{-1}$. We pick ϵ small enough that these ϵ-neighborhoods do not overlap. An example of such a region D is depicted below.

FIGURE 2. The Region D.



In D, the function f does not vanish, so by Cauchy's integral formula, we have

$$\int_{\partial D} d\left(\log f(z)\right) = \int_{\partial D} \frac{f'(z)}{f(z)} = 0$$

We see that ∂D consists of various parts:

- the horizontal line, from $\frac{1}{2} + iY$ to $-\frac{1}{2} + iY$ (the boundary of ϵ-neighborhood of ∞),

- the two vertical lines, ω to $-\frac{1}{2}+iY$ and $\omega+1$ to $\frac{1}{2}+iY$ (possibly with some ϵ neighborhoods removed),
- the circular arc from ω to $\omega+1$ (possibly with some ϵ-neighborhood removed),
- the boundaries of the ϵ-neighborhoods for each zero P.

In this last case, the total angle around $P \in \mathbb{H}/\Gamma(1)$ is 2π if P is not an elliptic fixed point (this either consists of a full circle around P, if P lies in the interior of $\mathcal{F}$, or of two half-circles if P lies on the boundary of $\mathcal{F}$ and is not ω, $\omega+1$ or i). At the points ω, $\omega+1$ and $i \in \mathbb{C}$, the integral of f only encircles a net angle of $2\pi/3$, $\pi/3$ and π, respectively. Thus for the single point in $\mathbb{H}/\Gamma(1)$ corresponding to the orbit of ω, we have a net angle of $\pi/3$, while for the single point corresponding to i we cover a net angle of π.

We now examine the various contributions to the integral. The two vertical lines give 0 as they are in opposite orientations and f is periodic. The horizontal line yields $-2\pi i\mathrm{ord}_\infty(f)$, which can be seen as follows. Say $\mathrm{ord}_\infty(f)=n$. We have

$$f(z)=a_nq^n+a_{n+1}q^{n+1}+\cdots$$

where $q=e^{2\pi iz}$. Thus $dq=2\pi ie^{2\pi iz}dz$ or $\frac{1}{2\pi i}\frac{dq}{q}=dz$. Thus:

$$d(\log f)=\frac{1}{f}\frac{d}{dz}(f)dz=\frac{2\pi ina_nq^n+2\pi i(n+1)a_{n+1}q^{n+1}+\cdots}{a_nq^n+a_{n+1}q^{n+1}+\cdots}\frac{dq}{q}\frac{1}{2\pi i}=\left(nq^{-1}+g(q)\right)dq$$

where g is a holomorphic function of the complex variable q. As z ranges from $\frac{1}{2}+iY$ to $-\frac{1}{2}+iY$, we see that q ranges one clockwise rotation around the circle of radius $e^{-2\pi Y}$. Thus, by Cauchy's theorem, the net contribution of this integral is $-2\pi in=-2\pi i\mathrm{ord}_\infty(f)$.

By Cauchy's theorem the contribution from the deleted ϵ-neighborhoods is $2\pi i\mathrm{ord}_P(f)$ for $P\in\mathbb{H}/\Gamma(1)$ not an elliptic fixed point (including those nonsingular $P\in\partial\mathcal{F}$). For these points $n_P=1$, so, vacuously, the integral contributes $2\pi i\mathrm{ord}_P(f)=\frac{1}{n_P}2\pi i\mathrm{ord}_P(f)$. For P an elliptic fixed point the net integral is also $\frac{1}{n_P}2\pi i\mathrm{ord}_P(f)$. This is because a sum of n_P equal copies of this integral corresponds (by the transformation symmetry of f) to an integral around a complete circular neighborhood of one of these singular points in the complex plane. An integral around such a complete circle, by Cauchy's theorem, yields $2\pi i\mathrm{ord}_P(f)$; hence the original integral around $P\in\mathbb{H}/\Gamma(1)$ contributes $\frac{1}{n_P}2\pi i\mathrm{ord}_P(f)$. Note that we are *subtracting* these contributions from the total integral, as these correspond to deleted neighborhoods.

Finally, we must compute the contribution of the arc from ω to $\omega+1$. We break up the integral into two halves: from ω to i and from i to $\omega+1$. We write

$$\begin{aligned}\int_\omega^{1+\omega}d(\log f(z))&=\int_\omega^i d(\log f(z))+\int_i^{1+\omega}d(\log f(z))\\&=\int_\omega^i d(\log f(z))-\int_\omega^i d\left(\log f\left(-\frac{1}{z}\right)\right)\\&=\int_\omega^i d(\log f(z))-\int_\omega^i d\left(\log\left(z^kf(z)\right)\right)\end{aligned}$$

We see that $d(\log(z^k f(z))) = kd\log(z) + d(\log f(z)) = \frac{k}{z} + d(\log f(z))$. Thus the two integrals involving $d\log(f)$ cancel, yielding

$$-\int_{\omega}^{i} kd\log(z) = \frac{\pi i k}{6}.$$

Putting all these contributions together we see that

$$-2\pi i \sum_{P\in\mathbb{H}/\Gamma(1)} \frac{1}{n_P}\mathrm{ord}_P(f) - 2\pi i\mathrm{ord}_\infty(f) + \frac{\pi i k}{6} = 0,$$

whence

$$\sum_{P\in\mathbb{H}/\Gamma(1)} \frac{1}{n_P}\mathrm{ord}_P(f) + \mathrm{ord}_\infty(f) = \frac{k}{12}.$$

□

As an immediate corollary, we see:

Corollary 2.3.2. The dimension of $M_k(\Gamma(1))$ is 0 for $k < 0$ and for k odd. For positive, even k, we have:

$$\mathrm{dim} M_k(\Gamma(1)) \leq \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & x \not\equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor & x \equiv 2 \mod 12. \end{cases}$$

Proof. Set $m = \lfloor \frac{k}{12} \rfloor + 1$ and choose m distinct points P_i, none of which are elliptic fixed points. Given any collection of modular forms $f_1, , f_{m+1}$, each of weight k, by elementary linear algebra we can find a linear combination of the f_i, say f, that vanishes at all these points. However, in this case the LHS of the Valence formula is strictly larger than the RHS, and the term at infinity and contributions from elliptic fixed points can only increase the RHS. Thus $f = 0$ identically, and the f_i have a nontrivial linear relation. Thus $\mathrm{dim} M_k(\Gamma(1)) \leq \lfloor \frac{k}{12} \rfloor + 1$. In the case of $k \equiv 2 \mod 12$, we can reduce the bound by one through divisibility considerations: the RHS of the valence formula must have a denominator of 6, with a numerator 1 more than a multiple of 6. This will require contributions from each of the elliptic fixed points; in fact, we need at least a double zero at ω and a single zero at i, together contributing $2/3 + 1/2 = 7/6$. Then we have $m - 1 \in \mathbb{Z}$ remaining zeroes to choose from. Repeating the previous argument, we see, in this case, that $\mathrm{dim} M_k(\Gamma(1)) \leq m - 1$.

□

Corollary 2.3.3. By the previous corollary, $\mathrm{dim} M_{12}(\Gamma(1)) \leq 2$. If $\mathrm{dim} M_{12}(\Gamma(1)) = 2$, and f and g are two linearly independent modular forms, the function f/g supplies an isomorphism (of Riemann surfaces) between $\overline{\mathbb{H}/\Gamma(1)}$ and $\mathbb{P}^1(\mathbb{C})$.

Indeed, by the valence formula, for all $(\lambda, \mu) \neq (0, 0)$, the weight 12 modular form $\lambda f - \mu g$ must have precisely one zero in $\overline{\mathbb{H}/\Gamma(1)}$. Thus the function $\psi = f/g$ takes every value in $\mathbb{P}^1(\mathbb{C})$ exactly once. This gives a bijective holomorphic map, establishing a Riemann surface isomorphism between $\overline{\mathbb{H}/\Gamma(1)}$ and $\mathbb{P}^1(\mathbb{C})$.

Finally, we note that the valence formula generalizes to other groups Γ. The only change is that the constant 1/12 in the RHS is replaced with

$$\frac{1}{4\pi}\mathrm{Vol}(\mathbb{H}/\Gamma)$$

where this volume is taken with respect to the Poincare metric on the upper half plane. The only other group Γ we will deal with is the congruence subgroup $\Gamma(2)$ (and the isomorphic group $\Gamma_0(4)$).

2.4. **Eisenstein Series and the Algebra $M_*(\Gamma(1))$.** We must now get to the task of constructing actual modular forms. Firstly, we introduce the "slash operator." Given a function on $\mathbb{H}$, k an integer, and $\gamma : \mathbb{H} \to \mathbb{H}$ a fractional linear transformation (say $\gamma(z) = (az+b)/(cz+d)$), the slash operator is a transformation of f is given by:

$$(f|_k\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

It follows from the cocycle condition (7) that $f|_k(\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2$ for γ_1 and $\gamma_2 \in \mathrm{SL}(2,\mathbb{R})$. Moreover, the slash operator of a function of subexponential growth is itself of subexponential growth. Therefore if $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ a discrete subgroup, Γ acts on the vector space of holomorphic functions on $\mathbb{H}$ which have subexponential growth via the slash operator. Modular forms of k are then simply the invariant subspace under this action.

In representation theory, an important technique used to construct a G-invariant vector is to symmetrize: given an arbitrary vector v one sums (or integrates) over its orbit. (Sometimes one averages so that the symmetrization operation fixes vectors that are already invariant.) Moreover, if one has a subgroup v_0 that is already invariant under some subgroup G_0 of G, then $g(v)$ depends only upon the left coset of g; indeed, $gg_0(v_0) = g(v_0)$ for all $g_0 \in G_0$. We can therefore choose left coset representatives g_i for G/G_0, and take the sum $\sum_i g_i(v_0)$ to get something G-invariant. This has the potential to reduce the complexity of the sum if we are lucky enough to start out with a vector v_0 that has some invariance; moreover, if G_0 is infinite then this method is imperative, as using the standard symmetrization procedure, $\sum_{g\in G} g(v_0)$, gives us an infinite sum of copies of v_0 as g ranges over all elements of G_0.

We shall apply this technique to construct modular forms of weight k on $\Gamma(1)$. Note that the slash operator is a *right* Γ action. We will apply the symmetrization technique to the simplest possible function: $f(z) = 1$. Note that f is already invariant under the slash operator $(1|_kT^a)$ for all k and a with T the translation operator defined in (9). (Note that the slash operator for other $\gamma \in \Gamma(1)$ introduces negative powers of $(cz+d)$, so $f(z) = 1$ is not invariant under those transformations.) The (positive and negative) powers of T are precisely the elements of $\Gamma(1)$ that fix infinity, so we call this subgroup Γ_∞. As we have seen, there are no modular forms of odd weight; so for all values of k we will be dealing with, $(1|_k(-I)) = 1$. Thus we lose nothing by using the modular group, $\mathrm{PSL}(2,\mathbb{Z}) := \overline{\Gamma(1)}$, as opposed to $SL(2,\mathbb{Z}) = \Gamma(1)$. In this case, the stabilizer of infinity $\overline{\Gamma_\infty}$ consists solely of powers of T. So we shall sum the function $f(z) = 1$ over all *right* coset representatives of $\overline{\Gamma(1)}/\overline{\Gamma_\infty}$. We must now turn our attention to finding a natural family of coset representatives for $\overline{\Gamma(1)}/\overline{\Gamma_\infty}$.

Firstly, for a Möbius transform $\gamma : z \mapsto (az+b)/(cz+d)$, we have $\gamma(\infty) = a/c$. Thus γ fixes infinity precisely when $c = 0$, and since we are working in $\mathrm{PSL}(2,\mathbb{Z})$, we know that the matrices with $c = 0$ correspond to powers of T. If we multiply an arbitrary 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the left by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, the result is

$$\begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$$

which has the same bottom row as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So let us examine the bottom rows of matrices in $\overline{\Gamma(1)}$. Firstly, such a pair of integers is necessarily prime or else the determinant would be divisible by their common factor; conversely, by Bezout's lemma, for any relatively prime pair c and d, there exists a and b such that $ad - bc = 1$.

We claim that the bottom row of a matrix uniquely determines its right coset in $\overline{\Gamma(1)}/\overline{\Gamma_0}$. We have already seen one direction of this. Conversely, if two (determinant 1, integer) matrices have the same bottom row (consisting of a pair of relatively prime prime integers) then we can find an n such that $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ left multiplies one to the other. Indeed: if we have $ad - bc = 1$ and $a'd - b'c = 1$, then $(a-a')d = (b-b')c$. Since d and c have no common factors, $\frac{a-a'}{c} = \frac{b-b'}{d}$ is an integer, and this is the value of n we desire.

Thus, we can now define:

$$E_k(z) := \sum_{\gamma \in \Gamma(1)/\Gamma_\infty} 1|_k\gamma = \sum_{\gamma \in \overline{\Gamma(1)}/\overline{\Gamma_\infty}} 1|_k\gamma = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}, \tag{13}$$

where the 1/2 factor is added because (c,d) and $(-c,-d)$ represent the same matrix in $\Gamma(1)$ (the the same coset in $\Gamma(1)/\Gamma_\infty$, since Γ_∞ contains -1). These are called *Eisenstein series.*

These series are absolute convergent for $k > 2$, as the number of pairs (c,d) such that $N \leq |cs+d| \leq N+1$ is the number of lattice points in in the annulus bounded by circles radius $N+1$ and N; as the area of this annulus is $\pi(N+1)^2 - \pi N^2 = O(N)$, we see that the series is majorized by $\sum_{N=1}^\infty N^{1-k}$. For odd k, (13) gives 0, as the (c,d)-term cancels out with the $(-c,-d)$-term; we would expect this sort of behavior lest E_k for odd K it give us a modular form of odd weight. $E_k(z)$ gives us modular forms for $k = 4$ and above.

Now, we can prove the following theorem:

Theorem 2.4.1. The ring $M_*(\Gamma(1))$ is freely generated by the modular forms E_4 and E_6.

Proof. We first show that E_4 and E_6 are algebraically independent. We first demonstrate that E_4^3 and E_6^2, both of weight 12, cannot be multiples of one another.[12] For if they were, and say $\lambda E_4^3 = E_6^2$, for $\lambda \neq 0$, then we can let $f(z) = E_6/E_4$, a modular function of weight 2, which satisfies $f^2 = \lambda E_4$ and $f^3 = \lambda^{-1} E_6$. As f is a function whose square is holomorphic, f itself cannot have poles and so must be holomorphic. But this contradicts the fact that there are no holomorphic modular forms of weight 2 – a consequence of Corollary 2.3.2.

We claim that any two modular forms f_1 and f_2 of the same weight, which are not multiples of one another, are algebraically independent. For if $P(X,Y) \in \mathbb{C}[X,Y]$ is a polynomial such that $P(f_1, f_2) = 0$, then, if we examine weights, we find that each homogeneous component must vanish identically.[13] Let P_d be the weight d component of P. We have $0 = P_d(f_1, f_2)/f_2^d := p(f_1/f_2)$. But p can only have a finite number of roots, so that f_1/f_2 is a constant. We thus see that E_4^3 and E_6^2, and therefore also E_4 and E_6, are algebraically independent.

Now, the independence of E_4 and E_6 implies that for each integer $k > 0$ and nonnegative integers a, b such that $4a + 6b = k$, the various monomials $E_4^a E_6^b$ of weight k are linearly independent. This gives a lower bound on the dimension $M_k(\Gamma(1))$ for each k: the number of solutions to $4a + 6b = k$ for nonnegative integers a and b. Let us call this number r_k. Note that $r_k = 0$ for k odd. For even k we see that $r_0 = 1$, $r_2 = 0$ and $r_4 = r_6 = r_8 = r_{10} = 1$ (corresponding to E_4, E_6, E_4^2, and E_4E_6, respectively).

Now, write $k = 12l + r$, with $0 \leq r < 12$. Each of the l sets of 12 can be either written as $4+4+4$ or $6+6$. We can choose to decompose each 12 into either $0, 1 \ldots, l$ distinct sums of the form $4 + 4 + 4$, correspondingly decomposing the remaining 12's into $l, l-1 \ldots, 1, 0$ sums of the form $6+6$. This gives a total of $l+1$ ways to write $12l$ as a positive combination of 4's and 6's. The remaining r can be written in terms of 4 or 6 as described above, unless $r = 2$. In this case, subtract 6, giving us $12l + 2 - 6 = 12(l-1) + 8$, and use the same technique (i.e., write $12(l-1)$ as a sum of l distinct combinations of $4+4+4$ and $6+6$ and the remaining 8 as $4+4$). This shows that $\lfloor k/12 \rfloor + 1 \leq r_k$ if $k \not\equiv 2 \mod 12$, and $\lfloor k/12 \rfloor \leq r_k$ if $k \equiv 2 \mod 12$. However, $r_k \leq \dim M_k(\Gamma(1))$, and that Corollary 2.3.2 says that exactly these lower bounds for r_k are *upper* bounds for M_k. Therefore, all are equal, proving that $M_*(\Gamma(1))$ is precisely the algebra freely generated by E_4 and E_6.

□

[12]It is important to note first that neither $E_4(z)$ nor $E_6(z)$ are identically 0. An quick-and-dirty way to see this is to note that $E_4(z) \to 1$ as $\mathrm{Im}(z) \to \infty$, which follows trivially from setting $q = 0$ in the q-expansions of E_4 and E_6 that are given in the next section.

[13]By "homogeneous component" we mean all terms of a given weight; that is, the sum of monomials of the form $cE_4^a E_6^b$ where $4a + 6b = d$. For, indeed, if $P(E_4(z), E_6(z)) = 0$ for all z, then substitute $z \mapsto -1/z$. This gives us $\sum_d P_d(E_4(z), E_6(z))z^d = 0$ where P_d is the weight d component of P. We can once again make the substitution $z \mapsto -1/z$, giving $\sum_d P_d(E_4(z), E_6(z))z^{2d} = 0$, and indeed, arguing by induction, we see that $\sum_d P_d(E_4(z), E_6(z))z^{kd} = 0$ for all k. If the largest d such that P_d appears in P is n, then we see that the column vector $P_d(E_4(z), E_6(z))$ for $d = 1, \ldots, n$ is annihilated by the Vandermonde matrix with rows $1, x^k \ldots, x^{kn}$ for $k = 0, \ldots, n-1$. As this Vandermonde matrix is nonsingular for all but a finite number of $z \in \mathbb{H}$, each $P_d(E_4(z), E_6(z))$ must vanish all all but a finite number of points in $\mathbb{H}$, whence they must vanish everywhere.

Corollary 2.4.2. The inequalities of Corollary 2.3.2 are exact. That is, for positive, even k:

$$\dim M_k(\Gamma(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & x \not\equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor & x \equiv 2 \mod 12. \end{cases}$$

Corollary 2.4.3. (Tongue-in-cheek.) If k is a nonnegative, even integer, the number of ways of writing $k = 4a + 6b$ for nonnegative integers a and b is exactly

$$r_k = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & x \not\equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor & x \equiv 2 \mod 12. \end{cases}$$

Proof. We have already shows the the LHS is less than or equal to r_k. Corollary 2.3.2 implies the reverse inequality.[14]

There are various other common forms for Eisenstein series. Another form, written G_k, is given by

$$G_k(z) := \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

We can see that:

$$\begin{aligned} \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k} = &\left(\sum_{d=1}^{\infty} \frac{1}{d^k}\right) \left(\frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^k}\right) \\ =&\zeta(k) E_k(z) \end{aligned}$$

where $\zeta(k)$ is the Riemann zeta function. A final normalization of the Eisenstein series is given by:

$$\mathbb{G}_k(z) := \frac{(k-1)!}{(2\pi i)^k} G_k(z).$$

Each of these three variants has its own advantages; we will tend, following Viazovska, to use $E_k(z)$ in our work on sphere packings.

[14]This must be one of the most complicated proofs of this wholly elementary arithmetic fact. There are, of course, much more direct ways of seeing the reverse inequality without appealing to the dimension of weight k modular forms and the Valence formula.

2.5. **Fourier Expansions of Eisenstein Series.** We now give the Fourier expansion of the Eisenstein series:

Theorem 2.5.1. The Fourier expansion of the Eisenstein series $\mathbb{G}_k(z)$ for $k > 2$ even, is given by:

$$\mathbb{G}_k(z) = -\frac{B_k}{2k} + \sum_{k=1}^{\infty} \sigma_{k-1}(n) q^n \tag{14}$$

where B_k is the kth Bernoulli number, and $\sigma_{k-1}(n)$ is the sum of the $(k-1)$st powers of the divisors of n.

The Bernoulli numbers, which arise throughout mathematics but were first discovered in the context of the polynomials giving the sum of the first n kth powers ("Faulhaber's formula"), are usually defined via the generating series: $x/(e^x - 1) = \sum_{k=0}^{\infty} B_k x^k / k!$.

Proof. We begin with a well-known identity of Euler (see [35; 40]),

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z+n} = \pi \cot(\pi z) \tag{15}$$

We first observe that:

$$\pi \cot(\pi z) = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = -\pi i \frac{1+q}{1-q} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r \right)$$

with $q = e^{2\pi i z}$, as usual. (This equality holds for all $z \in \mathbb{H}$, where $|q| < 1$, so that we can expand the geometric series.) Substitute into (15), and differentiate this expression $k-1$ times, killing the nettlesome $\frac{1}{2}$ term (for $k \geq 2$). Divide by $(k-1)! \cdot (-1)^k$. Note that $\sum_{n \in \mathbb{Z}} 1/(z+n)^k$ converges absolutely for $k \geq 2$, whence we can get rid of the limiting operation on the LHS of (15). So, for $k \geq 2$, and $z \in \mathbb{H}$ we have:

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot(\pi(x))) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r \tag{16}$$

This identity is called "Lipschitz's formula," though it was certainly known to Eisenstein in his work on elliptic functions. For an account of this theory, which derives all the formulae discussed in this section in a highly novel way, see chapters 1 and 2 of [40].

We return to our formula for $G_k(z)$, and assume k is even. We break off the $m = 0$ terms:

$$
\begin{aligned}
G_k(z) &= \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^k} + \frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ m\neq 0}} \frac{1}{(mz+n)^k} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}
\end{aligned}
$$

whence we can apply (16) and Euler's formula for $\zeta(k)$, $k > 0$ even, in terms of the Bernoulli numbers[15]:

$$
\begin{aligned}
G_k(z) &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty} r^{k-1}q^{mr} \\
&= \frac{(2\pi i)^k}{(k-1)!}\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right).
\end{aligned}
$$

Multiplying by the normalizing factor yields:

$$
\mathbb{G}_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n
$$

as desired.

□

We can now explicitly compute:

$$
\begin{aligned}
\mathbb{G}_4(z) &= \frac{1}{240} + q + 9q^2 + 28q^3 + \cdots, \\
\mathbb{G}_6(z) &= -\frac{1}{504} + q + 33q^2 + 244q^3 + \cdots, \\
\mathbb{G}_8(z) &= \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots,
\end{aligned}
$$

and similarly:

$$
\begin{aligned}
E_4(z) &= 1 + 240q + 2160q^2 + 6720q^3 + \cdots, \\
E_6(z) &= 1 - 504q - 1663233q^2 - 122976q^3 - \cdots, \\
E_8(z) &= 1 + 480q + 61920q^2 + 1050240q^3 + \cdots.
\end{aligned}
$$

[15]Euler's formula is: $\zeta(k) = -i^k B_k (2\pi)^k/(2\cdot k!)$, for k even.

Note that we can immediately supply nontrivial algebraic relations among the various Eisenstein series. For example, E_4^2 is a modular form of weight 8; the space of such modular forms is 1 dimensional, generated by E_8. Thus E_4^2 is a multiple of E_8; as both have a constant term of 1 in their Fourier expansion, they are in fact equal. Similarly, we can show that $E_4E_6 = E_{10}$, and that $E_6E_8 = E_4E_{10} = E_{14}$. [16]

2.6. **The Non-Modular E_2 and Hecke's Trick.** In considering Eisenstein series of weight k, we had to assume $k > 2$ to have absolute convergence. However, we can let $k = 2$ in (14), to give a possible "$\mathbb{G}_2$" and corresponding E_2 and G_2. That is, we let:

$$\mathbb{G}_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

and

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n. \tag{17}$$

For $z \in \mathbb{H}$, this series converges absolutely and exponentially. Note that by construction this function is periodic. Moreover, the proof of Theorem 2.5.1 suffices to show that

$$G_2(z) = \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2}. \tag{18}$$

This double sum does not converge absolutely, so it is no longer true that the sum can be arbitrarily re-ordered. However, G_2 does possess some properties similar to those of modular forms.

Theorem 2.6.1. Let $z \in \mathbb{H}$ and $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Then:

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$

Proof. The proof that we shall present, following Zagier, involves "Hecke's trick." It is based on the observation that while (18) does not converge absolutely, it almost does – if we increase the exponent to $2(1+\epsilon)$, then we will have absolute convergence. We introduce:

$$G_{2,\epsilon}(z) = \frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}\frac{1}{(mz+n)^2|mz+n|^{2\epsilon}}. \tag{19}$$

This series converges absolutely, so applying the transformation γ, an rearranging terms, gives us:

[16]Equating the Fourier coefficients of these series give nontrivial identities for sums of powers; e.g., $\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m) = (\sigma_7(n) - \sigma_3(n))/120$.

$$G_{2,\epsilon}\left(\frac{az+b}{cz+d}\right) = (cz+d)^2|cz+d|^{2\epsilon}G_{2,\epsilon}(z).$$

We next show that $\lim_{\epsilon\to 0} G_{2,\epsilon}(z) = G_2(z) - \frac{\pi}{2y}$, where $y = \operatorname{Im}(z)$. This implies that the following (non-holomorphic) functions all transform like modular forms:

$$\begin{aligned} G_2^*(z) &= G_2(z) - \frac{\pi}{2y}, \\ E_2^*(z) &= E_2(z) - \frac{3}{\pi y}, \\ \mathbb{G}_2^*(z) &= \mathbb{G}_2(z) + \frac{1}{8\pi y}. \end{aligned}$$

We shall let:

$$I_\epsilon(z) = \int_{-\infty}^{\infty} \frac{dt}{(z+t)^2|z+t|^{2\epsilon}}.$$

When $\epsilon > 0$, we have:

(20)

$$\begin{aligned} G_{2,\epsilon}(z) - \sum_{m=1}^{\infty} I_\epsilon(mz) = &\sum_{n=1}^{\infty}\frac{1}{n^{2+2\epsilon}} \\ &+ \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\left(\frac{1}{(mz+n)^2\,|mz+n|^{2\epsilon}} - \int_n^{n+1}\frac{dt}{(mz+t)^2\,|mz+t|^{2\epsilon}}\right) \end{aligned}$$

The first sum converges absolutely and locally uniformly for $\epsilon > -\frac{1}{2}$. Let us examine the magnitude of the second sum. Note that it is essentially the difference between a sum and an integral of the corresponding function. By the mean value theorem, given any differentiable function f bounded on the interval $n \le t \le n+1$, the function $f(t) - f(n)$ is bounded by $\max_{n\le x\le n+1}|f'(x)|$. Letting $f(t) = 1/\left((mz+t)^2\,|mz+t|^{2\epsilon}\right)$, and noting that we can rewrite the function in the double summand as $\int_n^{n+1}\left(\frac{1}{(mz+n)^2|mz+n|^{2\epsilon}} - \frac{1}{(mz+t)^2|mz+t|^{2\epsilon}}\right)dt$, we see that each term in the double sum is $O\left(|mz+n|^{-3-2\epsilon}\right)$, and so the double sum converges absolutely and locally uniformly for $\epsilon > -\frac{1}{2}$.

Thus, the $\lim_{\epsilon\to 0} G_{2,\epsilon}$ exists and is obtained by evaluating (20) at $\epsilon = 0$. This equals $G_2(z)$ by (18) and since the various summands contributed by the integral telescope. Now we shal turn to the question of evaluating the sum $\sum_{m=1}^{\infty} I_\epsilon(mz)$ as $\epsilon \to 0$.

Observe that for $\epsilon > -1/2$:

$$\begin{aligned} I_\epsilon(x+iy) &= \int_{-\infty}^{\infty} \frac{dt}{(x+t+iy)^2\left((x+t)^2+y^2\right)^\epsilon} \\ &= \int_{-\infty}^{\infty} \frac{dt}{(t+iy)^2(t^2+y^2)^\epsilon} \\ &= \frac{1}{y^{1+2\epsilon}} \int_{-\infty}^{\infty} \frac{dt}{(t+i)^2(t^2+1)^\epsilon} \end{aligned}$$

where the last integral is obtained by making the substitution $y \mapsto ty$ (and $dt \mapsto ydt$). We already see that the mysterious term term $\sum_{m=1}^{\infty} I_\epsilon(mz)$ depends only on $y = \text{Im}(z)$. Let us define:

$$I(\epsilon) := \int_{-\infty}^{\infty} \frac{dt}{(t+i)^2(t^2+1)^\epsilon}.$$

We see that $\sum_{m=1}^{\infty} I_\epsilon(mz) = I(\epsilon)\zeta(1+2\epsilon)/y^{1+\epsilon}$ for $\epsilon > 0$. Now, $I(0) = 0$, and

$$I'(0) = \int_{-\infty}^{\infty} -\frac{\log(t^2+1)}{(t+i)^2} dt = \left(\frac{1+\log(t^2+1)}{t+i} - \tan^{-1} t \right)\Bigg|_{-\infty}^{\infty} = -\pi$$

via integration by parts (in particular, letting $u = \log(t^2+1)$ and $v = \frac{1}{t+i}$ whence $dv = -\frac{1}{(t+i)^2}$). As $\epsilon \to 0$, we see that $\zeta(1+2\epsilon) = \frac{1}{2\epsilon} + O(1)$. Thus $I(\epsilon)$ approaches $-\frac{\pi}{2y}$.

Therefore, the function $G_2^*(z) = G_2(z) - \frac{\pi}{2\text{Im}(z)}$, transforms like a modular form. It is called an "almost holomorphic modular form." We see that

$$(cz+d)^2 \left(G_2(z) - \frac{\pi}{2\text{Im}(z)} \right) = G_2^* \left(\frac{az+b}{cz+d} \right) = G_2 \left(\frac{az+b}{cz+d} \right) - \frac{\pi}{2\text{Im}\left(\frac{az+b}{cz+d}\right)}.$$

Recalling (3), and noting[17] that $-(cz+d)^2 + |cz+d|^2 = -2ic\text{Im}(z)(cz+d)$ we see that

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$

□

We note that E_2 transforms like:

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) - \frac{6i}{\pi} c(cz+d) \tag{21}$$

and that in particular,

[17]We know already that $G_2(z)$ is holomoprhic, so we know in advance that there must be some way of writing $G_2\left(\frac{az+b}{cz+d}\right) - G_2(z)$ in trms of the single complex variable z. Thus the fact that, miraculously, the non-holomorphic factors of $\text{Im}(z)$ cancel out should not come as a surprise.

$$E_2\left(-\frac{1}{z}\right) = z^2 E_2(z) - \frac{6i}{\pi} z. \tag{22}$$

We shall use this identity in our work on sphere packings.

2.7. **The Discriminant Function Δ.** The Discriminant function, $\Delta(z)$, is one of the most important modular forms. It is usually defined:[18]

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \tag{23}$$

for all $z \in \mathbb{H}$, the terms of the product approach 1 exponentially, whence the product converges everywhere and defines a holomorphic function on all of $\mathbb{H}$. We shall now show that it is modular.

Theorem 2.7.1. $\Delta(z)$, as defined by (23), is a modular form of weight 12 on $\Gamma(1)$.

Clearly Δ satisfies (11), we must show that it also satisfies (12). Now, let us take the logarithmic derivative of Δ:

$$\begin{aligned}
\frac{d}{dz} \log \Delta(z) &= 2\pi i - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \cdot 2\pi i \\
&= 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}\right) \\
&= 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n\right) \\
&= 2\pi i \, E_2(z),
\end{aligned}$$

the penultimate equality following from expanding $q^n/(1 - q^n)$ as geometric series and collecting terms.[19] Now, we observe that:

$$\frac{1}{2\pi i} \frac{d}{dz} \log \left(\frac{\Delta\left(\frac{az+b}{cz+d}\right)}{(cz + d)^{12} \Delta(z)} \right) = \frac{1}{(cz + d)^2} E_2\left(\frac{az + b}{cz + d}\right) - \frac{12}{2\pi i} \frac{c}{cz + d} - E_2(z) = 0$$

by (21), whence we see that $(\Delta|_{12}\gamma)(z) = C(\gamma)\Delta(z)$, for C some constant depending on γ. Note, however, that the function $C : \Gamma(1) \to \mathbb{R}$ is a homomorphism of groups from $\Gamma(1)$ to

[18] Of course it is not at all obvious *a priori* that Δ is a modular form, and one might reasonably wonder where such a thing could have come from. Indeed, Δ is most natural in the context of elliptic curves (which also explains its name). We have not discussed the relationship between modular forms and elliptic curves, so we can offer little further enlightenment on this point, but see [3] or [41].

[19] This kind of an expansion is called a Lambert series.

$\mathbb{C}^*$. That is, C defines a character on $\mathrm{SL}(2,\mathbb{Z})$. To see that this character is trivial (that is, $C(\gamma) = 1$ for all $\gamma \in \Gamma(1)$), we need only check that this is the case on the generators, S and T. However, Δ is periodic, whence $C(T) = 1$. To show $C(S) = 1$, we observe that $\Delta(-1/z) = C(S)\Delta(z)$ and evaluate at $z = i$. Now $\Delta(i)$ cannot be 0, as we can see from the product expansion. Therefore $C(S) = 1$.

This establishes Theorem 2.7.1.

□

We now note that, as a modular form of weight 12, Δ must be a linear combination of E_4^3 and E_6^2. Note $\Delta(z) = q + \cdots$, $E_4(z)^3 = 1 + 720q + \cdots$, and $E_6(z)^2 = 1 - 1008q + \cdots$. So we have:

$$\Delta(z) = \frac{1}{1728}\left(E_4(z)^3 - E_6(z)^2\right) \tag{24}$$

Say h is an arbitrary modular form of a weight k (k even), and let $a_0 := h(\infty)$ (i.e., the constant term of the Fourier expansion). Given some pair of numbers a and b such that $4a+6b = k$ (which, we have seen, is doable for all even $k > 4$), then $h - a_0E_4^aE_6^b$ is a modular form that vanishes at infinity, i.e., that has a 0 constant term in its Fourier expansion. we divide this modular form by Δ. The resulting function, $(h - a_0E_4^aE_6^b)/\Delta$, transforms like a modular form of weight $k-12$. Moreover, all of its negative-indexed q-coefficients all vanish, so it satisfies the necessary growth conditions. Iterating this process gives us a constructive algorithm to express modular forms in terms of E_4 and E_6; in fact, applying induction and Corollary 2.3.2. (to supply us with the base case) this gives an independent proof that $M_*(\Gamma(1)) = \mathbb{C}[E_4, E_6]$ (i.e., Theorem 2.4.1).

The product expansion shows us that Δ does not vanish anywhere on $\mathbb{H}$. In fact, we can already see this by the valence formula, which shows that can have no roots anywhere in $\mathbb{H}$ since it vanishes to order 1 at ∞. As $\dim M_{12}(\Gamma(1)) = 2$, we see that Δ is the unique modular form of weight 12 with expansion beginning $q + \cdots$. This is perhaps a more natural definition than (23). A modular form that vanishes at ∞ is called a "cusp form" – cusp forms are extremely important in the theory of modular forms.

The Fourier expansion of $\Delta(z)$ is of great interest. We define $\tau(n)$ to be the coefficient of q^n; $\tau(n)$ is called the Ramanujan tau function and it is one of the most intriguing functions in number theory. The expansion begins:

$$\Delta(z) = 1 - 24q + 252q^2 - 1472q^3 + 4830q^4 + \cdots$$

Ramanujan observed, among other remarkable properties of this sequence, that $\tau(n)$ is *multiplicative*; i.e., $\tau(mn) = \tau(m)\tau(n)$ for m and n relatively prime. Moreover, Ramanujan gave an estimate of the growth rate of these coefficients, which suggested that they grow unexpectedly slowly. This conjecture was resolved by Pierre Deligne, who both proved that Ramanujan's conjecture followed from the (very difficult) Weil conjectures (concerning the number of solutions of polynomial equations over finite fields), and then completed the proof of the necessary part of the Weil conjectures.

The last thing we shall define in this section is the modular j-invariant, which will supply us with an explicit isomorphism of the kind called for in Corollary 2.3.3.[20] Indeed, we define:

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 169884q + 21493760q^2 + \cdots \tag{25}$$

This function, though failing to abide by the necessary growth rate conditions (it has a first order pole at ∞, and so is not a modular form per se), defines an absolute invariant under the action of the modular group: $j((az+b)/(cz+d)) = j(z)$ for all a, b, c and d integers such that $ad - bc = 1$.

2.8. **Theta Series.** Let $\Lambda \subset \mathbb{R}^n$ be a lattice. We define the function $\theta_L : \mathbb{H} \to \mathbb{C}$ by:

$$\theta_L(z) = \sum_{x \in \Lambda} q^{\langle x,x \rangle/2} \tag{26}$$

This is called the θ-series of the lattice Λ. The theta series of a lattice is an important invariant of the lattice. Moreover, as we shall see, the Poisson sum formula shows that the theta series are closely related to modular forms – indeed, in the case of unimodular lattices, the theta series is in fact a bona fide modular form (though sometimes of non-integral weight).

The simplest lattice is $\mathbb{Z} \subset \mathbb{R}$. The corresponding theta function is called the Jacobi theta function. We will write:[21]

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + \cdots .$$

Theorem 2.8.1. $\theta(z)$ is a periodic function on $\mathbb{H}$, and satisfies

$$\theta\left(-\frac{1}{4z}\right) = \sqrt{\frac{2z}{i}}\theta(z) \tag{27}$$

for all $z \in \mathbb{H}$.

Proof. That $\theta(z)$ is periodic is true as by definition it has a q series. The second follows from the Poisson summation formula. Recall that this says that the if $f : \mathbb{R} \to \mathbb{C}$ is a well-behaved function which vanishes sufficiently quickly at infinity, then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$.[22] Indeed, if we define $g(x) := \sum_{n\in\mathbb{Z}} f(x+n)$, then g is a manifestly periodic function (given sufficiently fast decay at infinity), so the RHS can be written as a Fourier

[20]Though the important point about corollary 2.3.3. is that any ratio of non-proportional modular forms of weight 12 yields an isomorphism from $\overline{\mathbb{H}/\Gamma(1)}$ to $\mathbb{P}^1(\mathbb{C})$.

[21]Note that this is strictly speaking the theta series of the lattice $\sqrt{2}\mathbb{Z}$, not $\mathbb{Z}$ itself, whose theta series would be a sum of terms $q^{n^2/2}$. We wish to follow Zagier's convention.

[22]Recall that the Fourier transform, which will figure importantly in the Cohen-Elkies linear programming bound, is given by $\widehat{f}(x) := \int_{-\infty}^{\infty} f(y)e^{2\pi i y x}dy$.

series: $g(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$, with $c_n = \int_0^1 g(x)e^{-2\pi inx}dx = \widehat{f}(-n)$. Thus $\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \widehat{f}(-n)e^{2\pi inx}$. Evaluating this expression at $x = 0$ yields the Poisson sum formula.

Now, let $f(x) = e^{-\pi tx^2}$, with $t > 0$. We have:

$$\widehat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi tx^2+2\pi ixy}dx = \frac{e^{-\pi y^2/t}}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\pi u^2}du$$

where $u = \sqrt{t}\left(x - \frac{iy}{t}\right)$. Now, we shift the path of integration to the real line: observe that, as $\mathrm{Re}(u) \to \infty$, the integrand $e^{-\pi u^2}$ approaches 0 uniformly as $\mathrm{Re}(u) \to \infty$; also note that that $e^{-\pi z^2}$ is holomoprhic everywhere (so that shifting the contour does not cross any poles). Thus the integral reduces to the usual integral of a Gaussian, and we get:

$$\widehat{f}(y) = \frac{e^{-\pi y^2/t}}{\sqrt{t}}.$$

Therefore, for $t > 0$, we have:

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t}. \tag{28}$$

Thus we see that (27) holds for $z = it/2$, with $t > 0$. However, both sides of (27) define a holomoprhic function on $\mathbb{H}$. The two functions agree on the positive imaginary axis, hence they agree for all $\mathbb{H}$.

□

The two symmetries of θ, that is, $z \mapsto z+1$ and $z \mapsto -1/(4z)$ (which we view as $z \mapsto \frac{0z-1/2}{2z+0}$, so that the transformation corresponds to an element of $\mathrm{SL}(2,\mathbb{R})$), generate a subgroup of $\mathrm{SL}(2,\mathbb{R})$. This subgroup, as we shall see, is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ and contains a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Thus $\theta(z)$ is thus a modular form "of weight 1/2" – which for our purposes we shall define as a function whose square is a modular form of weight 1.

The underlying congruence subgroup for θ is $\Gamma_0(4)$. It is interesting to note that θ has an "extra" symmetry, not contained in $\Gamma_0(4)$; namely, $z \mapsto -1/(4z)$. This is a special example of a "Fricke involution." In general, we can extend the group $\Gamma_0(N)$ by the element $W_N = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Note that $W_N^2 = -1$, which acts like the identity on $\mathbb{H}$ (hence "involution") and that W_N normalizes $\Gamma_0(N)$. Thus the group generated by W_N and $\Gamma_0(N)$ is given by $\Gamma_0(N) \cup \Gamma_0(N)W_N$. It is often written $\Gamma_0^+(N)$.

In the case we are dealing with, $N = 4$, the group $\Gamma_0^+(4)$ is actually generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and W_4. (This is *not* true for general N.) We shall record this as:

Theorem 2.8.2. The group $\Gamma_0^+(4)$ is generated by T, the translation matrix, and W_4, the Fricke involution.

Proof. We define

$$\widetilde{T} := W_4 T W_4^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$

We shall show that T and $\widetilde{T}$ generate the image of $\Gamma_0(4)$ in $\mathrm{PSL}(2, \mathbb{Z})$. This implies that, up to a possible difference in sign, every element in $\Gamma_0(4)$ is in $\langle T, \widetilde{T} \rangle$. As $W_4^2 = -I$, we see that $\langle T, W_4 \rangle = \Gamma_0^+(4)$. Now, say that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. The idea of the argument is to multiply by T or $\widetilde{T}$ so as to decrease the quantity $a^2 + b^2$. Let us call the sum of the squares of the top row of a matrix the "gauge" of the matrix.

We see that a must be odd, else the determinant of γ would be even. Thus $|a| \neq 2|b|$. If $|a| < 2|b|$, then one of $b + a$ or $b - a$ is less than b in absolute value. Thus either γT or γT^{-1} has a strictly smaller gauge. If $|a| > 2|b|$ and $|b| \neq 0$, then $a + 4b$ or $a - 4b$ has a strictly smaller absolute value than a, whence either $\gamma \widetilde{T}$ or $\gamma \widetilde{T}^{-1}$ has a strictly smaller gauge. We can decreasing the gauge of our matrix by multiplying on the left by $T^{\pm 1}$ or $\widetilde{T}^{\pm 1}$ until $b = 0$. This matrix must, up to sign, be a power of $\widetilde{T}$. Thus completes our argument.

□

We now introduce two other important series, which we shall call, following Zagier, θ_M and θ_F. These are given by:

$$\theta_M(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + \cdots \tag{29}$$

$$\theta_F(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{n^2} = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots \tag{30}$$

An application of the Poisson sum formula as in Theorem 2.8.1 will show that these are both modular forms of weight 1/2 on $\Gamma_0(4)$, like θ. Moreover, we have the "Jacobi identity":

$$\theta(z)^4 = \theta_M(z)^4 + \theta_F(z)^4. \tag{31}$$

This expresses an equality among weight 2 modular forms on $\Gamma_0(4)$. This identity can be proved by showing, through arguments analogous to those laid out for $\Gamma(1)$ in this chapter, that the space of modular forms of weight 2 on $\Gamma_0(4)$ is 2 dimensional. Thus we need only verify that 1 coefficient of the q expansion of each side of (31) agrees with the other; indeed, both start $1 + 8q + \cdots$.

The triplet of θ, θ_M and θ_F are closely related to a similar triplet of modular forms on $\Gamma(2)$. This is due to the fact that there is an exceptional isomorphism between $\Gamma_0(4)$ and $\Gamma(2)$, which is given by conjugation by the element

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

which, it is worth noting, is in GL$(2,\mathbb{R})$, not SL$(2,\mathbb{R})$. This corresponds to the doubling transformation; thus a modular form f on $\Gamma_0(4)$ can be turned into a modular form on $\Gamma(2)$ by taking $f(z/2)$. We therefore have the triplet of modular forms of weight $1/2$ on $\Gamma(2)$:

$$\theta_{00}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}, \tag{32}$$

$$\theta_{01}(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 z}, \tag{33}$$

$$\theta_{10}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{\pi i n^2 z}. \tag{34}$$

Thus $\theta_{ij}^4(z)$ each define modular forms of weight 2 on $\Gamma(2)$. Inspection of the q expansions show that these modular forms satisfy:

$$\theta_{00}^4(z+1) = \theta_{01}^4(z), \tag{35}$$

$$\theta_{01}^4(z+1) = \theta_{00}^4(z), \tag{36}$$

$$\theta_{10}^4(z+1) = -\theta_{10}^4(z); \tag{37}$$

and an application of the Poisson sum formula yields:

$$\theta_{00}^4\left(-\frac{1}{z}\right) = -\theta_{00}^4(z), \tag{38}$$

$$\theta_{01}^4\left(-\frac{1}{z}\right) = -\theta_{10}^4(z), \tag{39}$$

$$\theta_{10}^4\left(-\frac{1}{z}\right) = -\theta_{01}^4(z). \tag{40}$$

We also have the Jacobi identity:

$$\theta_{00}^4(z) = \theta_{01}^4(z) + \theta_{10}^4(z). \tag{41}$$

These functions and the above identities will figure importantly in our work on sphere packing. We also have the following theorem, which we will state without proof, as the argument is essentially the same as the one we have already seen for $\Gamma(1)$.

Theorem 2.8.3. The space of modular forms on $\Gamma(2)$ is freely generated by θ_{00} and θ_{01}, or by any pair of the triplet θ_{00}, θ_{01} and θ_{10}.

Identities (32)–(40) imply that the modular form $\frac{1}{256}\theta_{00}^8\theta_{01}^8\theta_{10}^8$ is cusp form of weight 12 on $\Gamma(1)$ with q expansion beginning $q + O(q^2)$. This shows that

(42) $$\Delta(z) = \frac{1}{256}\theta_{00}^8\theta_{01}^8\theta_{10}^8.$$

We end this section by briefly returning to theta series associated to multidimensional lattices; in our case, of course, we want to know the theta series of E_8 and of Λ_{24}.

We see via an application of the Poisson sum formula and the unimodularity (i.e., self-duality) of E_8 that the corresponding theta series is a modular form of weight 4 on $\Gamma(1)$. As the constant term is 1, we see, in fact, that the theta series of E_8 is exactly $E_4(z)$; that is:

(43) $$\Theta_{E_8}(z) = E_4(z) = 1 + 240q + \cdots .$$

The Leech lattice has a noteworthy theta series, too. We see via Poisson summation, and self-duality of Λ_{24} that Λ_{24} has a theta series given by a modular form of weight 12 on $\Gamma(1)$. Moreover, we see that it begins $1 + O(q^2)$ as no vectors have a norm of 2 by Theorem 1.3.2. As the space of weight 12 modular forms is 2 dimensional, this determines the theta series:

(44) $$\Theta_{\Lambda_{24}}(z) = \frac{1}{12}\left(7E_4(z)^3 + 5E_6(z)^2\right) = E_{12}(z) - \frac{65520}{691}\Delta(z) = 1 + 196560q + \cdots .$$

The integrality of the coefficients of this q-series (which is manifest since they represent a count of lattice points) gives a fascinating congruence for the Ramanujan tau function; namely, that $\tau(n) - \sigma_{11}(n)$ is always divisible by 691. This extraordinary divisibility property was first observed by Ramanujan.

2.9. **Ramanujan's Derivative Identities.** The derivative of a modular form is not itself a modular form. For say we define the operator:

$$Df = f' := \frac{1}{2\pi i}\frac{df}{dz} = q\frac{df}{dq} = \sum_{n=1}^{\infty} n a_n q^n$$

for a modular form f. We have:

$$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2}f'(z) + \frac{k}{2\pi i}c(cz+d)^{k+1}f(z).$$

However, the operator ϑ_k defined by:

$$\vartheta_k f := f' - \frac{k}{12}E_2 f$$

maps $M_k(\Gamma(1))$ to $M_{k+2}(\Gamma_1)$. (This can be readily seen by an application of (22)). The operator ϑ_k is called the Serre derivative, and will often be notated simply ϑ (since usually will know the weight of the modular form to which we will apply it). This suggests that the extended ring $\widetilde{M}_*(\Gamma(1)) := M_*(\Gamma(1))[E_2] = \mathbb{C}[E_2, E_4, E_6]$ is of interest when we are dealing with issues of differentiation. We call this the ring of quasimodular forms on $\Gamma(1)$. We will soon discuss a more intrinsic way of defining a quasimodular form, which works for other groups, but for now we shall use the ad-hoc approach of saying that a quasimodular form on $\Gamma(1)$ is imply a polynomial in E_2 and other modular forms.

Theorem 2.9.1. The ring $\widetilde{M}_*(\Gamma_1)$ of quasimodular forms on Γ_1 is closed under differentiation. In particular, we have the "Ramanujan derivative identities":

$$E_2' = \frac{E_2^2 - E_4}{12}, \tag{45}$$

$$E_4' = \frac{E_2E_4 - E_6}{3}, \tag{46}$$

$$E_2' = \frac{E_2E_6 - E_4^2}{2}. \tag{47}$$

Proof. Observe that ϑE_4 and ϑE_6 are modular forms of weight 6 and 8, respectively. Hence they must be constant multiples of E_6 and E_4^2, respectively. By examining q series, we find that the constants are $-1/3$ and $-1/2$, respectively. To demonstrate the first identity, we differentiate (22), and find that $E_2' - \frac{1}{12}E_2^2$ is a modular form of weight 4. Therefore it is a multiple of E_4; examining the constant terms of the q series we find that the constant is $-1/12$. As these are the generators of the ring $\widetilde{M}_*(\Gamma_1)$, we see that $\widetilde{M}_*(\Gamma_1)$ is closed under differentiation.

□

We shall see that (46) will be of some use to us in our work on sphere packings. Indeed the function $(3E_4')^2 = (E_2E_4 - E_6)^2$ will appear as the numerator of Viazovska's +1-eigenfunction.

2.10. **Laplace Transforms of Modular Forms and Fourier Eigenfunctions.** There is an elegant way of constructing eigenfunctions of the Fourier transform via taking the Laplace transform of a modular form. This construction gives the starting point for Viazovska's work, and is the first strong suggestion that modular forms might be useful in constructing the 8 and 24-dimensional magic functions. Indeed, say we have $x \in \mathbb{R}^n$, and we define:

$$f(x) = \int_0^\infty g(t)e^{-\pi t|x|^2}dt.$$

We take the Fourier transform:

$$\widehat{f}(y) = \int_{\mathbb{R}^n}\left(\int_0^\infty g(t)e^{-\pi t|x|^2}dt\right)e^{-2\pi i\langle x,y\rangle}.$$

Now, assuming $g(t)$ is sufficiently well-behaved, then we can swap the two integrals. Recall that the n-dimensional Fourier transform of a Gaussian is given by:[23]

$$\mathcal{F}\left(e^{-\pi t|x|^2}\right) = t^{-\frac{n}{2}} e^{-\pi|x|^2/t}$$

Thus, we have:

$$\widehat{f}(y) = \int_0^\infty g(t) t^{-\frac{n}{2}} e^{-\pi|x|^2/t} dt.$$

Applying the transformation $t \mapsto 1/t$, we have

$$\widehat{f}(y) = \int_0^\infty g(1/t) t^{\frac{n}{2}-2} e^{-\pi|x|^2 t} dt.$$

Now, if $g(t) = \phi(it)$, where ϕ is a modular form of weight $2 - n/2$, then $g(1/t)t^{n/2-2} = \phi(i/t)t^{n/2-2} = i^{2-n/2}\phi(it) = i^{2-n/2}g(t)$. Thus, in this case the function $f(x)$ is an eigenfunction with eigenvalue $i^{2-n/2}$.

In the cases of $n = 8$ and $n = 24$, we have $2 - n/2 = -2$ and -10 respectively. This will provide us (in both cases) with a -1 eigenfunction.

3. The Cohn-Elkies Linear Programming Bound

In 2003, Henry Cohn and Noam Elkies published their so-called "linear programming bound," named in analogy with the linear programming bounds used to solve the 8 and 24-dimensional kissing number problem. 13 years later, Viazovska showed that this bound was optimal in 8 and 24 dimensions using an ingenious construction based on modular forms. Presently we explain the bound and some of its more immediate implications.

3.1. The Bound.

When we refer to "sufficiently well-behaved" functions we mean a function to which to which we can apply the Fourier transform, such that we can apply Fourier inversion and the Poisson Sum Formula. As it shall turn out, all the functions we will deal with belong to Schwartz Space, so more subtle real analysis will not play a role in our argument.

Theorem 3.1.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a suitably well-behaved, non-zero function of sufficiently fast decay. Assume f satisfies

$$f(x) \leq 0 \text{ for all } |x| > 1, \tag{48}$$

and

$$\widehat{f}(t) \geq 0 \text{ for all } t \in \mathbb{R}^n. \tag{49}$$

[23] See [23].

Then the center-density of any n-dimensional sphere packing cannot exceed

$$\frac{f(0)}{2^n \widehat{f}(0)}. \tag{50}$$

Proof. We will prove this bound for periodic packings, which in fact proves the result for all packings, for, as we have shown above, a periodic packing approximates the density an arbitrary packing arbitrarily well.

Suppose a sphere packing is given by translates of a lattice Λ by vectors $v_1, \dots, v_N$, such that $v_i - v_j \notin \Lambda$ for all $i \neq j$. We scale so that no two distinct sphere-centers have distance < 1 between them; that is, we make our spheres each have radius $1/2$. Then the center-density δ is given by:

$$\delta = \frac{N}{2^n |\Lambda|}.$$

From the Poisson summation formula, we have:

$$\sum_{x \in \Lambda} f(x+v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{-2\pi i \langle v, t \rangle} \widehat{f}(t)$$

for $v \in \mathbb{R}^n$. We now sum on v ranging over the differences between the vectors v_i (including N terms of $v_i - v_i = 0$):

$$\begin{aligned}
\sum_{1 \le j,k \le N} \sum_{x \in \Lambda} f(x + v_j - v_k) &= \frac{1}{|\Lambda|} \sum_{1 \le j,k \le N} \sum_{t \in \Lambda^*} \widehat{f}(t) e^{-2\pi i \langle v_j - v_k, t \rangle} \\
&= \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \left(\widehat{f}(t) \sum_{1 \le j,k \le N} e^{-2\pi i \langle v_j - v_k, t \rangle} \right) \\
&= \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \left(\widehat{f}(t) \sum_{1 \le j,k \le N} e^{-2\pi i \langle v_j, t \rangle} e^{2\pi i \langle v_k, t \rangle} \right) \\
&= \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \left(\widehat{f}(t) \sum_{1 \le j \le N} e^{-2\pi i \langle v_j, t \rangle} \sum_{1 \le k \le N} e^{2\pi i \langle v_k, t \rangle} \right) \\
&= \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \left(\widehat{f}(t) \overline{\sum_{1 \le j \le N} e^{2\pi i \langle v_j, t \rangle}} \sum_{1 \le k \le N} e^{2\pi i \langle v_k, t \rangle} \right) \\
&= \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \widehat{f}(t) \left| \sum_{1 \le j \le N} e^{2\pi i \langle v_j, t \rangle} \right|^2
\end{aligned}$$

Every term in the final equation is non-negative, so we bound the sum below by the single term with $t = 0$. This gives us a lower bound of $\frac{N^2 \widehat{f}(0)}{|\Lambda|}$. On the LHS of the above, we see that $x + v_j - v_k$ is the vector from the sphere-center v_k to the sphere-center $x + v_j$. By assumption, this always has a magnitude ≥ 1, unless $v_k = x + v_j$. As the v_i represent distinct cosets of Λ, this latter situation occurs if and only if $x = 0$ and $v_j = v_k$. Thus the LHS sum can be given by an upper bound of $Nf(0)$, and we have:

$$Nf(0) \geq \frac{N^2 \widehat{f}(0)}{|\Lambda|}$$

or, equivalently,

$$\delta \leq \frac{f(0)}{2^n \widehat{f}(0)}$$

finishing the argument.

□

A careful examination of the above argument reveals that it is extremely wasteful: we are bounding all the terms outside the unit ball on the RHS above by 0 and all but one of the terms on the LHS below by 0. Indeed, for most almost all dimensions n, it is expected that even a theoretical "optimal function" for the Cohn-Elkies bound (that is, a function for which $1/2^n \cdot f(0)/\widehat{f}(0)$ is minimal[24]) might be substantially greater than the densest possible sphere packing. For equality of δ and $f(0)$ to occur – that is, for the bound to actually *equal* the density of the periodic packing packing whose centers are given by the union of Λ-translates of v_i – we should have

$$Nf(0) = \frac{N^2 \widehat{f}(0)}{|\Lambda|}$$

which, we see from the above argument, will follow if, say, $\widehat{f}(t)$ *vanishes* on all Λ^*, and f vanishes on all $x + v_j - v_k$, $x \in \Lambda$, $j \neq k$. [25]

In particular, if we conjecture that a lattice packing Λ is optimal, then, if we appeal to the Cohn-Elkies bound to prove optimality, it suffices to find an $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the following[26]:

1) $f(x) \leq 0$ for all $|x| > 1$, and $f(x) = 0$ for $x \in \Lambda$ with $x \neq 0$,

2) $\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^n$, and $\widehat{f}(x) = 0$ for all $x \in \Lambda$ with $x \neq 0$,

[24] The question of the existence of such an optimal function for a general dimension n appears to be open – that is, we do not know whether there exists a function f that attains the infimum of $f(0)/\widehat{f}(0)$ taken over all functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying (48) and (49).

[25] Conversely, if $Nf(0) = N^2\widehat{f}(0)/|\Lambda|$, then $f(t)$ *must* equal 0 for all $t \in \Lambda^*$ such that $\sum_{1 \leq j \leq N} e^{2\pi i \langle v_j, t\rangle} \neq 0$.

[26] Indeed, in the case of lattice packings these conditions are necessary and sufficient for the Cohn-Elkies bound to prove optimality. For, in the case of a lattice, we see that $\sum_{1 \leq i \leq N} e^{2\pi i \langle v_j, t\rangle} = e^{2\pi i \langle 0, t\rangle} = 1 \neq 0$. (See the previous footnote.) Thus for the Cohn-Elkies bound to prove optimality, we *must* find an f satisfying 1), 2) and 3); moreover, finding such an f suffices to prove optimality.

and

3) $f(0)$ and $\widehat{f}(0)$ satisfy

$$\frac{f(0)}{\widehat{f}(0)} = \frac{1}{2^n|\Lambda|}.$$

Note that conditions (48) and (49) and the estimate (50) are invariant under rotation. Thus if f is an optimal function (that is, gives us the minimal possible upper bound (50) for δ), then so are all precompositions $f \circ \rho$ for $\rho \in \mathrm{O}(n, \mathbb{R})$. In fact, we may average such an f over all such rotations ρ to obtain a new, rotationally symmetric optimal function: thus, without loss of generality, we may assume that f is radial.[27]

One might wonder whether this is, in fact, a misleading assumption to make. Say that a function f satisfies 1), 2) and 3) for a lattice Λ. If $x \in \mathbb{R}^n$ is an arbitrary point whose magnitude equals that of a vector $x_0 \in \Lambda$ (or $\in \Lambda^*$), then $f(x) = 0$ (respectively, $\widehat{f}(x) = 0$) of necessity: otherwise, by inequality 1), the symmetrized f would be strictly less than 0 for all x with $|x| = |x_0|$, and so cannot equal zero at x_0.[28] (Similarly, the symmetrized $\widehat{f}$ would be strictly greater than 0 for all x a vector with a magnitude equal to that of an element of the dual lattice, contradicting the vanishing of $\widehat{f}$ at every dual lattice vector.) Thus a non-radial f satisfying 1), 2) and 3) would need to display the rather strange behavior of vanishing on concentric spheres of radii equal to the magnitudes of Λ (and, likewise, $\widehat{f}$ would need to vanish on concentric spheres of radius equal to the magnitudes of Λ^*) *before* we apply rotational symmetrization. Thus searching for a radial f in those cases where we believe the magic function exists seems to be most plausible.

For radial functions $f : \mathbb{R}^n \to \mathbb{R}$, we rather abusively write $f(r)$ in lieu of the more accurate but cumbersome "$f(x)$ for some x with magnitude equal to r." Observe that if f is radial and satisfies 1), 2) and 3), then, for the signs of the radial $f : \mathbb{R} \to \mathbb{R}$ to work out correctly, the roots of f are rather constrained. In particular:

A) $f(1) = 0$, vanishing to odd order; with $f(r) = 0$ vanishing to even order for all $r > 1$ a lattice magnitude.[29]

B) $\widehat{f}(r) = 0$ for all $r > 0$ equal to the magnitude of a vector in Λ^*; with this vanishing occurring to even order for all such r.

C) $\widehat{f}(0) = 2^n|\Lambda| \cdot f(0) > 0$.

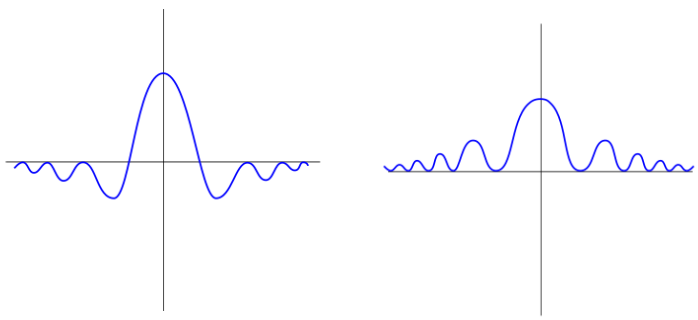
[27]We also see that the space of optimal functions is *convex* – that is, if f and g are both optimal functions in dimension n, then so is the average $\frac{f+g}{2}$.

[28]Technically speaking, it is possible for f to be negative on a set of measure 0 on the sphere of radius $|x_0|$; the argument we have presented shows that the set of all x such that $f(x) < 0$ (with the inequality strict) has measure 0 in the the sphere $|x| = |x_0|$ – otherwise symmetrization would lead to the average of f being strictly less than 0 everywhere on this sphere, which violates condition 1).

[29]Note that, for an optimal lattice sphere packing where every sphere has radius 1/2, the minimal nonzero magnitude of a vector in the lattice must be 1, else all points on the lattice are > 1 unit away from one another, and we can multiply by the reciprocal of this minimal distance and get a strictly denser lattice whose vectors still satisfy the property of being at least 1 unit away from each another.

This way, f starts positive, crosses the r axis at $r = 1$, and then bounces back down at every greater lattice magnitude (thereby staying negative). Meanwhile $\widehat{f}$ starts positive, hits zero at every magnitude of the dual lattice Λ^*, but always bounces back up (thereby remaining positive). As we shall see, the greater the order of vanishing that we insist that f take at the magnitudes of Λ (and likewise for $\widehat{f}$ and Λ^*), the more constrained f becomes. Thus it is simplest to assume that f has a first order zero at 1 with double-zeroes at all greater lattice magnitudes, while $\widehat{f}$ has double-zeroes for all nonzero dual lattice magnitudes. We call such an optimal function f, for which the Cohn-Elkies bound gives an upper bound equal to the density of a known sphere packing, a "magic function."

Figure 3. The graph of a "magic" function f (left) and its Fourier transform $\widehat{f}$ (right)



Famously, controlling f and its Fourier transform $\widehat{f}$ simultaneously is extremely difficult - this is the root of the Heisenberg uncertainty principle [23]. It should therefore not be terribly surprising that finding a function that satisfies A) – C) is very hard - indeed, such a function f associated to a lattice Λ does not appear to exist in almost all dimensions. Fascinatingly, however, Cohn and Elkies were able to find functions f that get extremely close to the hypothetical magic function in dimensions 2, 8 and 24. (They give a complete resolution to the trivial 1-dimensional sphere packing problem using the linear programming bound in their paper, which we shall soon describe). Viazovska found an explicit construction for the 8-dimensional magic function in 2016; a result almost immediately followed by an explicit construction for the magic function in 24 dimensions.

We conclude this section by giving an equivalent form of the Cohn-Elkies bound, which is easier to use for practical applications.

Theorem 3.1.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a suitably well-behaved, non-zero function of sufficiently fast decay. If f satisfies

1) $f(0) = \widehat{f}(0) > 0$.
2) $f(x) \leq 0$ for all $|x| > r$.
3) $\widehat{f}(t) \geq 0$ for all $t \in \mathbb{R}^n$.

Then the center-density of any n-dimensional sphere packing cannot exceed $(r/2)^n$.

Proof. Apply Theorem 3.1.1. to the function $g(x) = f(rx)$. Then $g(0) = f(0)$ and $\widehat{g}(0) = \frac{1}{r^n}\widehat{f}(0)$. Then $\delta \leq (r/2)^n \cdot f(0)/\widehat{f}(0) = (r/2)^n$.

□

3.2. **The Case of Dimension 1.** Cohn and Elkies give a rather entertaining proof that the lattice packing $\mathbb{Z} \subset \mathbb{R}^1$ is the densest packing of 1-spheres (i.e., intervals). This is, of course, entirely trivial, as 100% of space can be covered by tacking intervals together. But it is somewhat itneresting to see how the Cohn-Elkies method proves this vacuous result.

We shall let the magic function equal "the obvious" function that will force the roots to behave in accordance awith A) and B).

$$f(x) = \frac{1}{1-x^2}\left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

It is evident that this function's Fourier transform has compact support contained $[-1, 1]$. For we are taking the integral:

$$\widehat{f}(y) = \int_{-\infty}^{\infty} \frac{1}{1-x^2}\left(\frac{\sin(\pi x)}{\pi x}\right)^2 e^{-2\pi i x y} dx \tag{51}$$

and the integrand extends to a holomorphic function on the complex plane such that:

$$f(it) = \frac{1}{4\pi^2(1+t^2)t^2}\left(e^{2\pi t} - 2 + e^{-2\pi t}\right).$$

This is $O(e^{2\pi t})$ as $t \to \infty$, and $O(e^{-2\pi t})$, $t \to \infty$. Thus we can apply Cauchy's integral formula to (51) and push the contour out to $+i\infty$ if $y > 1$ and $-i\infty$ if $y < -1$, making the integral arbitrarily small. Thus for $|y| > 1$, $\widehat{f}(y) = 0$. For $|y| < 1$, we can check that:

$$\widehat{f}(y) = \frac{2\pi(1-|y|) + \sin(2\pi|y|)}{2\pi}.$$

The derivative of this function is $\cos(2\pi y) - 1$, so it is clear by elementary calculus that $\widehat{f}(y)$ is strictly decreasing from 0 to 1. As $\widehat{f}(y) = 1$ at $y = 0$ and $\widehat{f}(y) = 0$ at $y = 1$, we see that the sign of $\widehat{f}(r)$ is strictly positive for $0 < y < 1$. By symmetry, $\widehat{f}(y) > 0$ for $-1 < y < 0$. Thus we have $f(r) \leq 0$ for all $r > 1$, and $\widehat{f}(r) \geq 0$ always. Moreover, $f(r)$ satisfies conditions A) and B) corresponding to the packing $\mathbb{Z} \subset \mathbb{R}^1$, so this the $\mathbb{Z}$-packing is indeed optimal, with an impressive density of 100%.

While this argument is somewhat tongue-in-cheek, it does teach us some things, though rather more about the use of the Cohn-Elkies bound than about the not-too-onerous problem of packing 1-balls in $\mathbb{R}^1$ as tightly as possible. In particular, we see that a $\sin^2$ factor helps to force the zeroes into place while also behaving fairly nicely with respect to the Fourier transform. This insight is, as we shall see, critical to Viazovska's construction.

3.3. **Some Numerical Consequences of the Cohn-Elkies Bound.** We shall briefly discuss some of the fascinating computational consequences of the Cohn-Elkies bound. If we want to find a function $f : \mathbb{R}^n \to \mathbb{R}$ which will give us a good sphere packing bound in n dimesions, it is helpful to assume f to has the form:

$$f(x) = p(r^2)e^{-\pi r^2}, \quad |x| = r$$

for $x \in \mathbb{R}^n$, and try to optimize the polynomial p for increasing degrees.

The utility of this approach is that the Fourier transform of this function will be another function of the same form - that is, another polynomial in r^2 times $e^{-\pi r^2}$.

Indeed, following [9], say we let $p_j(x) = L_j^{n/2-1}(2\pi x)$, where L_j^α refers to the Laguerre polynomial of degree j and degree $\alpha = n/2 - 1$. Recall that the Laguerre polynomials are defined to to be the set of orthogonal polynomials with respect to the measure $x^{-\alpha}e^{-x}$ on $[0, \infty)$. Explicitly:

$$L_j^\alpha(x) = \frac{x^{-\alpha}e^x}{j!}\frac{d^j}{dx^j}(x^{\alpha+j}e^{-x}).$$

It is easy to verify that $p_j(|x|)^2 e^{-\pi|x|^2}$ is a radial eigenfunction of the Fourier transform of eigenvalue $(-1)^j$. That is, if $f(x) = p_j(r^2)e^{-\pi r^2}$, then

$$\widehat{f}(t) = (-1)^j p_j(|t|^2)e^{-\pi|t|^2}.$$

As the Laguerre polynomials p_j defined above form a basis for all polynomials, we see, by writing a general p in the basis p_j, that there exists a linear operator $\mathcal{T}$ on the space of polynomials in $r = |x|$, such that, if $f(x) = p(r^2)e^{-\pi r^2}$ for an arbitrary polynomial p, then

$$\widehat{f}(t) = \mathcal{T}(p)(|t|^2)e^{-\pi|t|^2}.$$

for all p. Note that $\mathcal{T}(p_j) = (-1)^j p_j$; the p_j for a diagonalizing basis for the operator $\mathcal{T}$. So writing p in the basis of Laguerre polynomials helps to simplify computations.

The original application of the linear programming bound, due to Cohn and Elkies themselves, went like this: we consider an arbitrary linear combination g of the odd-indexed Laguerre polynomials, $p_1, p_3, \ldots, p_{4m+3}$ for some value m. (Note that this is a -1 Fourier eigenfunction.) Then we insist upon p having m double roots at some set of values $z_1, \ldots, z_m$, as well as a root at 0. (Counting the number of degrees of freedom reveals that the polynomial g should be unique for a specified set of z_i). We try to adjust the z_i so as to minimize the value r of the smallest sign change of g. Then we solve for h, a linear combination of $p_0 \ldots p_{4m+2}$, with double roots at $z_1, \ldots z_m$, such that $g + h$ has a double root at r. We pick the signs of g and h so that $g + h$ is positive. We let $f = -g + h$, and note that $\widehat{f} = g + h$.

Now, our function f will be nonpositive outside of radius r, and will have an everywhere nonnegative Fourier transform. Moreover, $f(0) = \widehat{f}(0)$. We thus know from Theorem 3.1.2 that an n dimensional sphere packing has a center density bounded above by the number $(r/2)^n$.

Based on computer search, Cohn and Elkies found that as m increases the number r would decrease, apparently converging to an "optimal" bound in each dimension n. However, despite numerical evidence, there was (and still is) no guarantee that the optimization method will actually converge.

Moreover, for most n, the Cohn-Elkies bound was substantially larger than the densest known packing. However, in dimensions 2, 8 and 24, a miracle occurred – it appeared that the functions f computed in this way actually converged to above-described magic functions associated to the lattices A_2, E_8 and Λ_{24}. In dimension 8, the values z_i settled around $\sqrt{2k}$, $k \geq 2$ a positive integer while the number r approached the $\sqrt{2}$, exactly as needed. In 24 dimensions, the numbers z_i also approached $\sqrt{2k}$, $k \geq 3$, while the number r approached 2. [6] shows how close the numbers z_i are for $m = 11$:

Table 1. Values of r^2 and z_i^2 for $i = 1, \ldots, 11$

Case	r^2	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
8	2.0000	4.0000	6.0001	8.0004	10.0023	12.0095
24	4.000	6.001	8.003	10.010	12.029	14.0078

Case	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$	$i=11$
8	14.0448	16.1322	18.4054	20.8540	23.8869	27.7387
24	16.192	18.426	20.855	23.581	26.772	30.839

We see, however, that as the z_i increase, they begin to get larger than their expected value, and that this discrepancy increases with i.

One might wonder if optimizing over the roots is actually the most efficient way to approximate the magic function. Indeed, since we know already where the optimal function's roots "should" be in dimensions 8 and 24, why don't we just insist upon their correct location *a priori*? In other words, why don't we insist that, for 8 dimensions, say, our function f_k (a linear combination of the p_j) satisfies:

$$f_k(0) = 1$$
$$f_k(x) \text{ vanishes to order 1 at } |x| = r_1$$
$$f_k(x) \text{ vanishes to order 2 at } |x| = r_2, \ldots, r_k$$
$$\widehat{f}_k(x) \text{ vanishes to order 2 at } |x| = r_2, \ldots, r_k$$

There are $4k$ constraints, so we can assume f_k to be a combination of p_j with $j = 0, \ldots, 4k-1$. This method is rather promising; note that the constraint we have omitted is that $\widehat{f}(0) = 1$. Indeed, if this sequence of f_k converges to some f then $\widehat{f}(0) = 1$ is guaranteed by Poisson Summation. Unfortunately, as is discussed in [9], despite a very promising start for the 8 dimensional case, this method begins to stop working after $k = 40$ (which gives an upper bound of about 1.0000009656 times the actual density of E_8). Indeed, by $k = 120$, the bound has regressed to 1.5572034878 times the expected bound. In 24 dimensions, the failure is even more dramatic; the bound decreases to about 1.1082380574 times the expected (Leech lattice) density for $k = 20$, but by $k = 60$ gives us -3.7219923464 of the expected density, indicating that an unwanted sign change has occurred.

Indeed, we are fighting against the divergence phenomenon that we observed in the original Cohn-Elkies approximation method. As we find good approximations to the early roots of the magic function, the later roots of the approximation begin to diverge from their expected value. Strong-arming these roots into place, despite not technically overconstraining the system of polynomials, does not permit f_k to have enough breathing room. In turn, f_k fights back, eventually introducing extra (unwanted) sign changes.

[9] suggests an alternate, far less obvious method of root-forcing: for a given k the first two-thirds of the root locations (r_1 through $r_{\lfloor 2k/3 \rfloor}$ are left unchanged). Thereafter one perturbs the squares of the later roots by a quadratically growing amount so that the last root's square magnitude is off by a factor of 1.25. This *does* give good bounds, and appears to converge nicely.

Part of the problem with the naive method is that these more refined approximations seems to accumulate roots elsewhere in the complex plane. This phenomenon is reminiscent of partial products for theta series – for a given root of unity ω, the polynomials in q will vanish at $q = \omega$. Meanwhile, the theta function itself accumulates essential singularities along the unit circle. Similarly the actual magic function seems to have a (rather strange-looking) natural boundary (see [9] for a very interesting discussion of this phenomenon.) In constraining the function so as to only have the specified real roots, we fail to allow the function enough freedom to vanish at these other points.

This sort of problem exposes the fascinating difficulties one encounters in trying to control the vanishing behavior of f and $\widehat{f}$ simultaneously. Moreover, it shows how subtle the issue of convergence for these approximations might be.

Another striking result discovered through these computations was that, in the case of 8 and 24 dimensions, the magic functions appeared to be unique: various methods of approximate optimization, if they converged at all, converged to (what seemed to be) the same function. On the other hand there are already several known magic functions for dimension 1, and the magic function in dimension 2 (whose existence is still unknown) is not expected to be unique.

We conclude this section with a final note. Cohn and Miller were able to approximate the Taylor expansions of the (apparently unique) magic function f and its Fourier transform $\widehat{f}$ in dimensions 8 and 24. Mysteriously, they found that the quadratic coefficients appeared to be rational for both f and $\widehat{f}$:

$$f_8(x) = 1 - \frac{27}{10}|x|^2 + O(|x|^4) \tag{52}$$

$$\widehat{f}_8(x) = 1 - \frac{3}{2}|x|^2 + O(|x|^4) \tag{53}$$

in dimension 8 and

$$f_{24}(x) = 1 - \frac{14347}{5460}|x|^2 + O(|x|^4) \tag{54}$$

$$\widehat{f_{24}}(x) = 1 - \frac{205}{156}|x|^2 + O(|x|^4) \tag{55}$$

in dimension 24.

Alas, the higher coefficients exhibited no similar structure, preventing any effort to guess the function by its Taylor series. These conjectures would remain mysterious until the magic functions were constructed; moreover, they proved useful in helping to construct the magic functions in the first place.

4. Viazovska's Argument for Dimension 8

4.1. **Overview of the Construction.** We base this section in part on the excellent outline [4] of Viazovska's argument. We will try to give what one might call a "creation myth" for Viazovska's argument, explaining how one might, with some understanding of modular forms and their transformation behavior, come to construct the magic functions along the same lines as Viazovska.

We have seen from above that that finding an optimal function is finding a radial Schwartz function f such that f and $\widehat{f}$ have certain specified roots. The magnitudes of the vectors in E_8 are, as we have seen, the set $\{\sqrt{2k} : k = 1, 2, \dots\}$. Thus f must a simple root at $\sqrt{2}$ and double roots at each $\sqrt{2k}$, $k \geq 2$; and $\widehat{f}$ must have double roots at each $\sqrt{2k}$, $k \geq 1$. As we have pointed out, we do not expect that finding such an f will be particularly easy.

We also have the Cohn-Miller conjectures:

$$f_8(x) = 1 - \frac{27}{10}|x|^2 + O(|x|^4)$$
$$\widehat{f_8}(x) = 1 - \frac{3}{2}|x|^2 + O(|x|^4)$$

We shall begin by making a simplifying observation. We are controlling only the *roots* of f and $\widehat{f}$; we know, moreover, that f should be a radial function, which implies that $\widehat{\widehat{f}} = f$ (recall that generally $\widehat{\widehat{f}} = f(-x)$). A natural thing to try to do, then, is to break up f into *eigenfunctions* of the Fourier transform. In fact, making this assumption loses us no information, because we can *always* split a radial Schwartz function up into a $+1$ and -1 eigenfunction, just as every function can be split up into a sum of even and odd functions:

$$f_+ := \frac{f + \widehat{f}}{2} \text{ and } f_- := \frac{f - \widehat{f}}{2},$$

and indeed $f = f_+ + f_-$. Note that $\widehat{f_+} = \frac{\widehat{f}+f}{2} = f_+$ and $\widehat{f_-} = \frac{\widehat{f}-f}{2} = -f_-$, so that f_+ and f_- are indeed $+1$ and -1 eigenfunctions. Note that:

$$f_+ = 1 - \frac{21}{10}|x|^2 + O(|x|^4) \tag{56}$$

$$f_- = -\frac{3}{5}|x|^2 + O(|x|^4) \tag{57}$$

We might hope that we can force f_+ and f_- to have real roots at exactly $\{\sqrt{2k} : k = 1, 2, \ldots\}$; of course, we will need to show that no other real roots are introduced when we sum the two functions.

It is through the connection between modular forms and Fourier eigenfunctions that modular forms will become useful in our project to construct the magic function. However, Viazovska's actual construction – for both the $+1$ and the -1 eigenfunctions – is substantially trickier than the straightforward construction of taking the Laplace transform of a modular form of weight $2 - 8/2 = -2$. For, indeed, as shown above, if we let ψ be a modular form of weight -2, then its Laplace transform

$$f(r) := \int_0^\infty \psi(it) e^{-t\pi r^2} dt$$

satisfies $\widehat{f} = i^{-2} f = -f$. And, indeed, we can use a modular form ψ with an associated multiplier system (modular on some congruence subgroup) to obtain a $+1$ eigenfunction. It is thus natural to try to construct *each* eigenfunction as a Laplace transform of a modular form of some kind.

But this alone will not solve the problem of the magic function: there is no way to guarantee that the construction gives us the desired roots for f (and $\widehat{f}$). Viazovska's ingenious idea was simply to force f to have the desired roots by multiplying the above Laplace transform by a factor of $\sin^2(\pi r^2/2)$:

$$f(r) := \sin^2(\pi r^2/2) \int_0^\infty g(it) e^{-t\pi r^2} dt. \tag{58}$$

Now, $\sin^2(\pi r^2/2)$ has roots of multiplicity 2 at all points $r = \pm\sqrt{2k}$, $k \in \mathbb{Z}$, $k > 0$, and a root of multiplicity 4 at $r = 0$. Note that this is a little bit of overkill: we want roots of multiplicity merely 1 at $\pm\sqrt{2}$, and no vanishing at all at $r = 0$. So we will need to make sure that integral gives us a pole of order 4 at $r = 0$, and poles of order 1 at $r = \pm\sqrt{2}$. Moreover, we shall want it to have no poles anywhere else.

Let us briefly take note of what will be required of g to obtain such poles. By elementary calculus,[30]

$$\int_0^\infty t e^{-\pi r^2 t} dt = \frac{1}{\pi^2 r^4} \tag{59}$$

[30]The basic principles governing these formulae are that applying the Laplace transform to x^a (for $a > 0$) gives $1/x^{a+1}$ times a constant, and that if $F(x)$ has a Laplace transform $f(r)$, then $e^{ax}F(x)$ has a Laplace transform $f(r - a)$.

and

$$\int_0^\infty e^{2\pi t} e^{-\pi r^2 t} dt = \frac{1}{\pi(r^2-2)}. \tag{60}$$

If we consider this $g(t)$ to be something like a modular form, then we expect $g(t)$ to have a q-expansion. Then (60) suggests that $g(it)$ should have a q^{-1} term. On the other hand, (59) suggests that expanding $g(it)$ should have a "naked" t term (a term consisting of a constant times t with no factors of q), which suggests that $g(it)$ might expand into a product of polynomials in t and q series. Note that the t-term, which as we see from (59) gives rise to a $1/r^4$-term when we take the Laplace transform, controls the value of of f at 0 (once we multiply by $\sin^2(\pi r^2/2)$, which vanishes to order 4 at 0). By (56) this naked t-term must only come from the $+1$ eigenfunction.

4.2. **Constructing the $+1$ Eigenfunction.** Let us now consider the $+1$ eigenfunction case. Following Viazovska's notation we let $x \in \mathbb{R}^8$, and let $|x| = r$. Then let $a(x)$ be the $+1$ eigenfunction, which we shall construct as:

$$a(r) := \sin^2(\pi r^2/2) \int_0^\infty g(it) e^{-t\pi r^2} dt. \tag{61}$$

We shall now see what sorts of conditions on g are suggested by the ansatz $\widehat{f} = f$.

Note that the factor of $\sin^2$ now messes up the eigenfunction property of the Laplace transform. However, there is hope, as the function $\sin^2(\pi r^2/2)$ has an exponential expansion via Euler's formula:

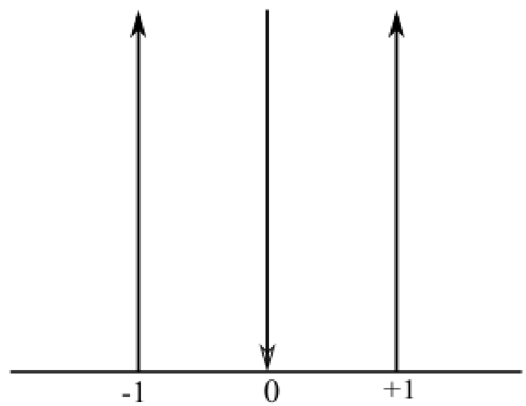
$$\sin^2(\pi r^2/2) = -\frac{1}{4}\left(e^{\pi i r^2} - 2 + e^{-\pi i r^2}\right).$$

This lets us do some contour shifting:

$$\begin{aligned}
a(r) &= \sin^2(\pi r^2/2) \int_0^\infty g(it) e^{-t\pi r^2} dt \\
&= -\frac{1}{4}\int_0^\infty \left(g(it)e^{-(t+i)\pi r^2} - 2g(it)e^{-t\pi r^2} + g(it)e^{-(t-i)\pi r^2}\right) dt \qquad (62)\\
&= -\frac{1}{4}\left(\int_i^{\infty+i} g(it+1)e^{-t\pi r^2}dt - 2\int_0^\infty g(it)e^{-t\pi r^2}dt + \int_{-i}^{\infty-i} g(it-1)e^{-t\pi r^2}dt\right) \\
&= \frac{i}{4}\left(\int_{-1}^{-1+i\infty} g(t+1)e^{i\pi r^2 t}dt - 2\int_0^{i\infty} g(t)e^{i\pi r^2 t}dt + \int_1^{1+i\infty} g(t-1)e^{i\pi r^2 t}dt\right) \qquad (63)
\end{aligned}$$

where we integrate vertically along the imaginary axis in the last integral. Firstly, note that g itself cannot be a modular form or else the integrand would be identically 0. So we shall have to be more clever to find g.

FIGURE 4. The path of integration seen in (63).



Now, let us apply the Fourier transform to this integral. Assuming g is sufficiently well-behaved that we can swap the order of integration, we can use the Fourier transform of an 8-dimensional Gaussian:

$$\mathcal{F}\left(e^{\pi i|x|^2 t}\right)(y) = t^{-4} e^{\pi i |y|^2\left(-\frac{1}{t}\right)}.$$

This will replace t by $-1/t$ and multiply the whole expression by t^{-4} in each integrand; we also, for convenience, bring the factor of $i/4$ to the other side. We have:

$$\begin{aligned} -4i\,\widehat{a}(r) = &\int_{-1}^{-1+i\infty} g(t+1)t^{-4}e^{i\pi r^2(-1/t)}dt - 2\int_0^{i\infty} g(t)t^{-4}e^{i\pi r^2(-1/t)}dt \\ &+ \int_1^{1+i\infty} g(t-1)t^{-4}e^{i\pi r^2(-1/t)}dt \end{aligned}$$

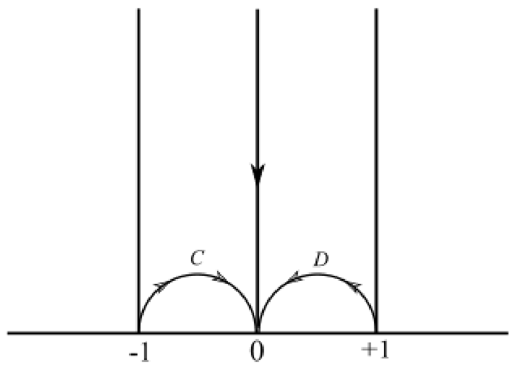
Substituting $t \mapsto -1/t$, $dt \mapsto t^{-2}dt$ we obtain:

$$\begin{aligned} -4i\,\widehat{a}(r) = &\int_C g\left(-\frac{1}{t}+1\right)t^2 e^{i\pi r^2 t}dt - 2\int_{i\infty}^{0} g\left(-\frac{1}{t}\right)t^2 e^{i\pi r^2 t}dt \\ &+ \int_D g\left(-\frac{1}{t}-1\right)t^2 e^{i\pi r^2 t}dt. \end{aligned} \tag{64}$$

where C and D are the transformed contours as depicted below.

Before we proceed let us note some of important aspects of the problem that we have encountered. Firstly, we see that if g is to have some kind of modular properties, there will be no escaping the substitutions $t \mapsto t+1$ and $t \mapsto -1/t$ – the first coming from the contours given by the $\sin^2$ factors and the second coming from applying the Fourier transform to a

FIGURE 5. The path of integration seen in (64).



Laplace transform. These, as was shown above, generate all of $\Gamma(1)$, so we should suspect that g is modular in some way on the full modular group $\Gamma(1)$.

Next, we might hope that by insisting that g have some modular properties, we can make this integral *manifestly* equal to the integral in (63). But it is far from clear how to do so, because circles C and D do not appear as contours in (63).

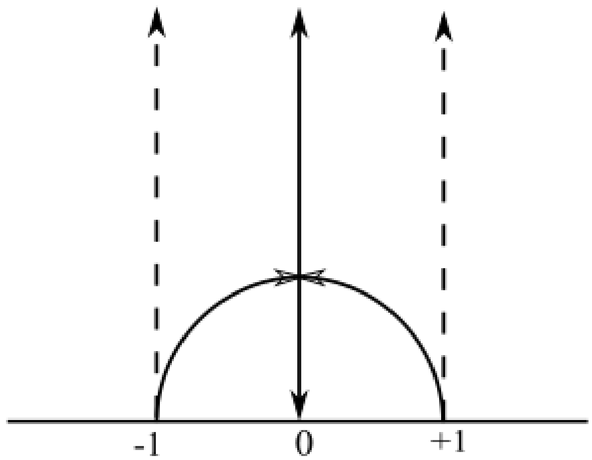
A more natural way to proceed would be to deform the contours found in (63) from the outset, so that the new contours are invariant under $t \mapsto -1/t$. To this end, the imaginary line iy is already invariant under the transformation $t \mapsto -1/t$. Thus we can keep the iy contour as it is. The most natural candidate for the $-1+it$ contour is to integrate along the boundary of the complex unit circle from -1 to i, and then from i to $i\infty$ along the imaginary axis; likewise, for the contour $1+iy$, we first integrate along the unit circle from 1 to i and then from i to $i\infty$ along the imaginary axis. The two quadrants of the unit circle in $\mathbb{H}$ are exchanged by the map $t \mapsto -1/t$, and the two halves of the imaginary axis (those iy with $y > 1$ and iy with $y < 1$) are exchanged by the transformation $t \mapsto -1/t$.

For the deformed path to not change the value of the integral, it is important that there be no poles of g anywhere inside the region lying above the unit circle and two vertical lines of real part -1 and $+1$. If we imagine that g is modular in some way under the action of $\Gamma(1)$, then g should have no poles anywhere in $\mathbb{H}$ as the this region contains the fundamental domain of $\Gamma(1)$.

Let us now write out the integral in (63) with these new contours:

$$\begin{aligned}-4ia(r) = &\int_{-1}^{i} g(t+1)e^{i\pi r^2 t}dt + \int_{i}^{i\infty} g(t+1)e^{i\pi r^2 t}dt \\ &+ \int_{1}^{i} g(t-1)e^{i\pi r^2 t}dt + \int_{i}^{i\infty} g(t-1)e^{i\pi r^2 t}dt \\ &- 2\int_{0}^{i\infty} g(t)e^{i\pi r^2 t}dt,\end{aligned}$$

FIGURE 6. The deformed path of integration of (63) invariant under $t \mapsto -1/t$.



or, grouping by contours:

$$(65)\qquad \begin{aligned} -4ia(r) = &\int_i^{i\infty} \left(g(t+1) - 2g(t) + g(t-1)\right) e^{i\pi r^2 t} dt \\ &+ \int_i^0 2g(t) e^{i\pi r^2 t} dt \\ &+ \int_{-1}^i g(t+1) e^{i\pi r^2 t} dt \\ &+ \int_1^i g(t-1) e^{i\pi r^2 t} dt. \end{aligned}$$

Now, let us apply the Fourier transform as above:

$$\begin{aligned} -4i\,\widehat{a}(r) = &\int_i^{i\infty} \left(g\left(t+1\right) - 2g\left(t\right) + g\left(t-1\right)\right) t^{-4} e^{i\pi r^2\left(-\frac{1}{t}\right)} dt \\ &+ \int_i^0 2g(t) t^{-4} e^{i\pi r^2\left(-\frac{1}{t}\right)} dt \\ &+ \int_{-1}^i g(t+1) t^{-4} e^{i\pi r^2\left(-\frac{1}{t}\right)} dt \\ &+ \int_1^i g(t-1) t^{-4} e^{i\pi r^2\left(-\frac{1}{t}\right)} dt. \end{aligned}$$

And, substituting $t \mapsto -1/t$ (as before, $dt \mapsto t^{-2} dt$):

$$-4i\,\widehat{a}(r) = \int_i^0 \left(g\left(-\frac{1}{t}+1\right) - 2g\left(-\frac{1}{t}\right) + g\left(-\frac{1}{t}-1\right)\right) t^2 e^{i\pi r^2 t} dt \tag{66}$$
$$+ \int_i^{i\infty} 2g\left(-\frac{1}{t}\right) t^2 e^{i\pi r^2 t} dt$$
$$+ \int_1^i g\left(-\frac{1}{t}+1\right) t^2 e^{i\pi r^2 t} dt$$
$$+ \int_{-1}^i g\left(-\frac{1}{t}-1\right) t^2 e^{i\pi r^2 t} dt.$$

Equating the integrands of (65) and (66) over the common contours gives us the following four functional equations for g:

$$g(t+1) - 2g(t) + g(t-1) = 2g\left(-\frac{1}{t}\right) t^2, \tag{67}$$

$$2g(t) = \left(g\left(-\frac{1}{t}+1\right) - 2g\left(-\frac{1}{t}\right) + g\left(-\frac{1}{t}-1\right)\right) t^2, \tag{68}$$

$$g(t+1) = g\left(-\frac{1}{t}-1\right) t^2, \tag{69}$$

$$g(t-1) = g\left(-\frac{1}{t}+1\right) t^2. \tag{70}$$

We can see at once that (67) and (68) are equivalent, as are (69) and (70); both via the substitution $t \mapsto -1/t$. Thus, we really have only two independent functional equations which we can say are given by (67) and (69). Let us examine (69), as it is the simpler of the two and more closely resembles the identity (12) for a garden-variety modular form.

Let us try to isolate $g(-1/t)$. Say we let $t+1 = -1/u$. Then $t = -1/u - 1$, whence $-1/t - 1 = -1/(u+1)$. We have:

$$g\left(-\frac{1}{u}\right) = g\left(-\frac{1}{u+1}\right)\left(-\frac{1}{u}-1\right)^2,$$

or, elegantly:

$$g\left(-\frac{1}{u}\right) u^2 = g\left(-\frac{1}{u+1}\right)(u+1)^2. \tag{71}$$

Formula (71) is truly remarkable: it says that the function $\phi(t) := t^2 g(-1/t)$ is periodic. So in fact (69), despite superficially resembling (12), is actually an analogue of (11), following from the periodicity of the function ϕ.

We see, therefore, that ϕ should be a function with a q-expansion. We should therefore hope to build ϕ out of modular forms. Note that $g(t) = \phi(-1/t)t^2$; this is the function whose Laplace transform we are taking to get the $+1$ eigenfunction.

Let us now turn to our other, more complicated, functional equation, (67). The important thing to note here is that the LHS is actually a second finite difference of $g(t)$; we now know that the RHS is equal to $2\phi(t)$, where ϕ is periodic. Thus (67) says that ϕ satisfies not the modular transformation property – that is, $\phi(-1/t)t^2 = \phi(t)$ – but"second finite difference modularity": $\frac{1}{2}\Delta^{(2)}[\phi(-1/t)t^2] = \phi(t)$. [31]

An ingenious idea of Viazovska's was to introduce the non-modular E_2 into the mix. Let us examine why this is a reasonable idea. Let us imagine that $\phi(t) = Q(E_4, E_6)$, where Q is some rational function. Then $t^2\phi(-1/t)$ is given by a sum of products of series in t times series in $q = e^{2\pi it}$. As discussed above, we want a naked t-term to show up in this expansion (in order to give a pole of order 4 when we take the Laplace transform; moreover, we saw that this naked t-term must come from the $+1$-eigenfunction). As all products $f = E_4^p E_6^q$ satisfy $f(-1/t) = t^k f(t)$ for some *even* power k (in fact $k = 4p + 6q$), we cannot expect to obtain a naked t^1-term from this method. However, $E_2(-1/t)$ *does* gives us a t term: $E_2(-1/t) = t^2E_2(t) - (6i/\pi)t$.

Let us imagine now that we take the second difference of $\phi(-1/t)t^2$, where ϕ is some rational function of E_2, E_4, and E_6. When we expand $Q(E_2(-1/t), E_4(-1/t), E_6(-1/t))t^2$, we shall obtain a sum of series in t times rational functions of E_2, E_4, and E_6. Applying a second finite difference to an expression of the form $t^k \cdot P(E_2(t), E_4(t), E_6(t))$ will simply yield $\Delta^{(2)}[t^k] \cdot P(E_2(t), E_4(t), E_6(t))$, by the periodicity of E_2, E_4 and E_6. [32] Note that $\Delta^{(2)}[t^k]$ is a polynomial of degree $k - 2$; in particular, it equals 2 for t^2, and 0 for both t and 1.

Thus, if we want $\Delta^{(2)}[\phi(-1/t)t^2] = 2\phi(t)$, the latter series contains no (nonzero) powers of t in its expansion, and so we should be able to expand:

$$\phi(-1/t)t^2 = t^2\phi(t) + t\psi_1(t) + \psi_2(t) \tag{72}$$

where ψ_1 and ψ_2 are two functions with q expansions (we can there assume the E_2, E_4, and E_6 without any t terms). Anything satisfying (72) then necessarily satisfies the relation $\Delta^{(2)}[\phi(-1/t)t^2] = 2\phi(t)$.

What does (72) tell us about ϕ as a rational function of E_2, E_4, and E_6? The first thing we should try to ascertain is the "weight" of ϕ. [33] But how can we get at the weight of a quasimodular form via its transformation behavior? We want to be able to think of $E_2(t)$ as being of "weight 2", and a general monomial $E_2^{k_2}E_4^{k_4}E_6^{k_6}$ as being of weight $2k_2 + 4k_4 + 6k_6$ (i.e., the weight of a product should be the sum of the weights, like it already is for modular forms). Now, $E_2(-1/t) = t^2E_2(t) + (6i/\pi)t$, and if we view this expression as a polynomial

[31] Here $\Delta^{(2)}$ refers to the second finite difference operation $f(z) \mapsto f(z+1) - 2f(z) + f(z-1)$, not to the discriminant function's square.

[32] More generally, if u is a periodic function, and v is any function, then *any* finite difference operator Δ^k satisfies $\Delta^k[uv] = u\Delta^k[v]$. More technically the finite difference operators commute with translation and thus act by endomorphisms on the module of functions over the ring of periodic functions.

[33] [41], (section 5.3) gives a detailed discussion of quasimodular forms in general, though he does not define weight like we do. We, however, do not need the full machinery of quasimodular forms in their generality.

in t only, then the highest power of t appearing in the expansion is t^2. We can think of "the highest power of t appearing when we substitute $t \mapsto -1/t$ and expand the result into quasimodular forms and powers of t" as a kind of definition of the weight. This agrees with the desideratum that weight of a product should be the sum of the corresponding weights: say we have some monomial $f(t) = E_2^{k_2} E_4^{k_4} E_6^{k_6}$. Then we have:

$$\begin{aligned} f(-1/t) &= \left(E_2(t)t^2 - \frac{6i}{\pi}t\right)^{k_2} \left(E_4(t)t^4\right)^{k_4} \left(E_6(t)t^6\right)^{k_6} \\ &= t^{2k_2+4k_4+6k_6} E_2^{k_2} E_4^{k_4} E_6^{k_6} + \cdots \end{aligned}$$

where the remaining terms contain strictly smaller powers of t. The same carries over to negative weights, that is, rational functions $P(E_2, E_4, E_6)/Q(E_4, E_6)$, where P and Q are both of a well-defined weight – we can simply take the difference between the weight of the numerator and the weight of the denominator.[34]

Now, (72) shows that

$$\phi(-1/t) = \phi(t) + t^{-1}\psi_1(t) + t^{-2}\psi_2(t) \tag{73}$$

so we see that ϕ should be of weight 0. Moreover, ϕ can have no poles in $\mathbb{H}$, so the natural thing to do is to divide something of weight 12 by Δ, which, as we have discussed, does not vanish in $\mathbb{H}$. We therefore write:

$$\phi(t) = \frac{P(E_2, E_4, E_6)}{\Delta}$$

where Δ is the modular discriminant $\frac{1}{1728}(E_4^3 - E_6^2)$ and $P(E_2, E_4, E_6)$ is a quasimodular form of weight 12. Thus P is a linear combination of the monomials E_4^3, E_6^2, $E_2E_4E_6$, $E_2^2E_4^2$. That is

$$\phi(t) = \frac{(AE_4^3 + BE_6^2) + (CE_4E_6)\,E_2 + (DE_4^2)\,E_2^2}{\Delta} \tag{74}$$

Then we have:

$$\phi(-1/t) = \frac{(AE_4^3 + BE_6^2)\,t^{12} + (CE_4E_6t^{10})\left(E_2t^2 - \frac{6i}{\pi}t\right) + (DE_4^2t^8)\left(E_2t^2 - \frac{6i}{\pi}t\right)^2}{t^{12}\Delta} \tag{75}$$

Thus, expanding (75), isolating the t^{-1} and t^{-2} terms and comparing to (73), we see that:

[34]When the denominator is infected by E_2, the transformation $t \mapsto -1/t$ does not merely scale the denominator but adds an additional term. Sums in a denominator do not distribute like sums in a numerator; the appearance of an additional term makes the above definition of weight unworkable.

$$\psi_1(t) = \frac{-\frac{6i}{\pi}CE_4E_6 - \frac{12i}{\pi}DE_4^2E_2}{\Delta} \tag{76}$$

$$\psi_2(t) = \frac{-\frac{36}{\pi^2}DE_4^2}{\Delta} \tag{77}$$

Now we can attempt to ascertain see what these constants ought to be. Firstly, note that:

$$\sin^2(\pi r^2/2) = \frac{\pi^2 r^4}{4} - \frac{\pi^4 r^8}{48} + \frac{\pi^6 r^{12}}{1440} - \frac{\pi^8 r^{16}}{80640} + O(r^{20}).$$

We also have that:

$$\sin^2(\pi r^2/2)\int_0^\infty \phi(i/t)(it)^2 e^{-t\pi r^2}dt = 1 - \frac{21}{10}r^2 + \cdots$$

from which we see that:

$$\int_0^\infty \phi(i/t)(it)^2 e^{-t\pi r^2}dt = \frac{4}{\pi^2 r^4} - \frac{42}{5\pi^2 r^2} + \cdots . \tag{78}$$

Now, substituting it into (72) gives us:

$$-t^2\phi(i/t) = -t^2\phi(it) + it\psi_1(it) + \psi_2(it), \tag{79}$$

which yields

$$a(r) = \sin^2(\pi r^2/2)\int_0^\infty \left(-t^2\phi(it) + it\psi_1(it) + \psi_2(it)\right) e^{-\pi r^2 t}dt. \tag{80}$$

The naked t term must come from the term $(it \cdot \text{Const})$ in $it\psi_1(it)$. We can thus see, looking once more at (59), that the constant term of $\psi_1(t)$ should be $-4i$ to yield the first term on the LHS of (78).

Now, let us expand (76). We have:

$$\begin{aligned}\psi_1(t) =& \frac{-6i}{\pi}C(1+240q+\cdots)(1-504q+\cdots)(q^{-1}+24+324q+\cdots)\\ &- \frac{12i}{\pi}\cdot D(1+480q+\cdots)(1-24q+\cdots)(q^{-1}+24+324q+\cdots)\end{aligned}$$

Here we have here two unknowns, C and D. As we have already noted, the constant term of the expansion must be $-4i$, which gives us one constraint. But, in fact, we also know that there can be no q^{-1} term in the q expansion of ψ_1: a q^{-1} term in ψ_2 would include an integral of the form $\int_0^\infty te^{2\pi t}e^{-\pi r^2 t}dt = \frac{1}{\pi^2(r^2-2)^2}$, which would eliminate the desired zeros at $r = \pm\sqrt{2}$. Hence we must have the two q^{-1}-terms cancel; i.e., $C = -2D$. Thus, we have two constraints, and we can solve for C and D:

$$C = -\frac{\pi}{1080}, \quad D = \frac{\pi}{2160}.$$

However, we also have a $1/r^2$-term in (78), which comes from the Cohn-Miller conjecture for the quadratic term of the Taylor expansion of the magic function f. [35] Recall that:

$$\int_0^\infty e^{-\pi r^2 t}dt = \frac{1}{\pi r^2}$$

which means that the second term on the LHS of (78) must come from the constant term of $\phi(i/t)(it)^2$. This constant term must be contributed by $\psi_2(it)$, and must be equal to $\frac{-42}{5\pi}$.

Expansion of (77) yields:

$$\begin{aligned}\psi_2 &= -\frac{36}{\pi^2}\cdot\frac{\pi}{2160}(1+480q+\cdots)(q^{-1}+24+324q+\cdots)\\ &= -\frac{36}{\pi^2}\cdot\frac{\pi}{2160}(q^{-1}+504+\cdots)\\ &= -\frac{1}{60\pi}q^{-1}-\frac{42}{5\pi}+\cdots\end{aligned}$$

Miraculously, this gives us precisely the constant term predicted by Cohn-Miller! This is a sure sign that our method is on the right track.

Let us now attempt to solve for A and B. The integral (80) cannot contain any $t^2e^{2\pi t}$ (coming from a q^{-1} in the q-expansion of $\phi(it)$ in $-t^2\phi(it)$) term because

$$\int_0^\infty -t^2e^{2\pi t}e^{-\pi r^2 t}dt = -\frac{2}{\pi^3(r^2-2)^3}.$$

which, as before, would give us an unwanted pole at $r = \pm\sqrt{2}$. Similarly, we want $\phi(it)$ to not posses a constant term in its q-expansion, which will gives us, from the $-t^2\phi(it)$ term, a summand of the form:

$$\int_0^\infty -t^2e^{-\pi r^2 t}dt = -\frac{2}{\pi^3 r^6}.$$

[35] So far in our constructions we have not assumed this conjecture – the constant term of 1 in the Taylor expansion $a(r)$ follows directly from the version of the linear programming bound in which we normalize our magic function and its Fourier transform so that both are equal to 1 at $r = 0$. The conjecture only gives us novel information in the quadratic term of the Taylor expansion.

which has a pole of too high an order at $r = 0$. Thus ϕ must have no q^{-1} term, nor any constant q-expansion term. These two constraints, given our knowledge of C and D, are sufficient to solve for A and B.

Indeed, expanding (74), we see that:

$$\begin{aligned}\phi(t) =& A(1 + 720q + \cdots)(q^{-1} + 24 + 324q + \cdots) \\ &+ B(1 - 1008q + \cdots)(q^{-1} + 24 + 324q + \cdots) \\ &- \frac{\pi}{1080}(1 - 24q + \cdots)(1 + 240q + \cdots)(1 - 504q + \cdots)(q^{-1} + 24 + 324q + \cdots) \\ &+ \frac{\pi}{2160}(1 + 480q + \cdots)(1 - 48q + \cdots)(q^{-1} + 24 + 324q + \cdots)\end{aligned}$$

which simplifies to $A + B - \pi/2160 = 0$; $744A - 984B - 41\pi/90 = 0$. The solutions of these are

$$A = 0, \quad B = \frac{\pi}{2160}.$$

Putting this all together, and factoring out $\pi/2160$, we get:

$$\phi(t) = \frac{\pi}{2160} \cdot \frac{(E_2E_4 - E_6)^2}{\Delta} \tag{81}$$

We shall have that $a(r) := \sin^2(\pi r^2/2) \int_0^\infty \phi(i/t)(it)^2 e^{-\pi r^2 t} dt$, with ϕ as defined by (81), gives us the +1 eigenfunction.[36]

4.3. **Constructing The -1 Eigenfunction.** We now wish to construct the -1-eigenfunction. We want a function of the form:

$$b(x) = \sin^2(\pi r^2/2) \int_0^\infty g(it) e^{-t\pi r^2} dt,$$

that satisfies $\mathcal{F}(b)(x) = -b(x)$. We can use exactly the same contour shifting technique that we used for the +1-eigenfunction to get some functional equations for our function $g(t)$. Proceeding exactly as we did in (62)-(66), we can force our function b to be a -1 eigenfunction by making g satisfy:

$$g(t+1) - 2g(t) + g(t-1) = -2g\left(-\frac{1}{t}\right)t^2 \tag{82}$$

[36]This is not the same form in which Viazovska writes the +1 eigenfunction. She lets $\varphi_{-4}(z) = E_4^2/\Delta$, $\varphi_{-2}(z) = -E_4E_6/\Delta$, and then lets $\phi_0 := \varphi_{-4}E_2^2 + 2\varphi_{-2}E_2 + j - 1728$. Recalling that $j = E_4^3/\Delta$, this is readily seen to be equivalent up to a constant scaling factor. Viazovska scales her ϕ_0 so that $\phi_0 = q^{-1} + \cdots$; we scale our ϕ_+ so that $\sin^2$ times the Laplace transform is exactly the +1 eigenfunction.

(83) $$2g(t) = -\left(g\left(-\frac{1}{t}+1\right) - 2g\left(-\frac{1}{t}\right) + g\left(-\frac{1}{t}-1\right)\right)t^2$$

(84) $$g(t+1) = -g\left(-\frac{1}{t}-1\right)t^2$$

(85) $$g(t-1) = -g\left(-\frac{1}{t}+1\right)t^2.$$

We can see that (82) and (83) are equivalent, as are (84) and (85) – both via the transformation $t \mapsto -1/t$.

Proceeding as before, we look at (84) and isolate an expression for $g(-1/t)$. Sending $t \mapsto -1/t - 1$, we have:

$$g\left(-\frac{1}{t}\right)t^2 = -g\left(-\frac{1}{t+1}\right)(t+1)^2$$

which shows us that the function $\phi(t) := g(-1/t)t^2$ satisfies:

(86) $$\phi(t+1) = -\phi(t).$$

Following the +1-eigenfunction case, we should expect ϕ to be a modular – or quasi-modular – form on some congruence subgroup of level 2. As shifting by 1 introduces a factor of -1, we can expect that ϕ has a Fourier series in $q^{1/2}$, containing only odd powers of $q^{1/2}$.

Now, let us consider the implications of (82). We note that it is equivalent to

$$\Delta^{(2)}[\phi(-1/t)t^2] = -2\phi(t)$$

where, as above, $\Delta^{(2)}$ is the second finite difference operator. Substituting $t \mapsto t+1$ in (82), adding it back to (82), and applying (86), we get $g(t+2) - g(t+1) - g(t) + g(t-1) = 0$. Thus $g(t+2) - g(t) = g(t+1) - g(t-1)$ is periodic (and therefore has a q-series), and $g(t) - g(t-1) = g(t+2) - g(t+1)$ is periodic with period 2 (and therefore has a $q^{1/2}$-series). To satisfy the difference equations, we see that we should let

$$\phi(-1/t)t^2 = g(t) = -\phi(t)/2 + \psi_1(t) + t\psi_2(t),$$

where ψ_1 and ψ_2 are both quasimodular forms on $\Gamma(2)$ with period 1. This implies that the "weight" of the quasimodular form must be 1, which is impossible, as the weights of quasimodular forms on $\Gamma(2)$ are all even.[37] Thus $\psi_2(t) = 0$. We see, therefore, that both

[37]See Zagier's treatment of quasimodular forms: section 5.3. of [41]; however, note that Zagier does not adopt our ad-hoc definition of weight. It turns out that in general, the ring of quasimodular forms (which we have not defined here) is generated by a single non-modular quasimodular form of weight 2, and the standard modular forms. In the case of $\Gamma(1)$, this means that the ring of quasimodular forms is precisely $\mathbb{C}[E_2, E_4, E_6]$. In this case, note that $-I \in \Gamma(2)$, whence all weights of modular forms and quasimodular forms are even.

$\phi(t)$ and $g(t) = \phi(-1/t)t^2$ should have $q^{1/2}$-expansions; we can expect that ϕ to not just be quasimodular, but actually modular and of weight -2.

Note that we have, in some ways, a simpler situation than that of the $+1$-eigenfunction: we do not need to worry about quasimodularity. The function whose Laplace transform we are taking, $g(t)$, is actually equal to a (negative weight) modular form.

Returning to the above integral, we see, by setting $g(t-1) = g(t+1)$ in (82), that $-2g(t) + 2g(t-1) = -2g(-1/t)t^2$, or simply:

$$g(-1/t)t^2 + g(t-1) = g(t). \tag{87}$$

Viazovska writes the three summands of (87) as $\psi_S = g(-1/t)t^2$ (which we have also called $\phi(t)$), $\psi_T = g(t-1)$ (which is also $g(t+1)$ by the 2-periodicity of g), and $\psi_I = g(t)$, respectively (S and T refer to the generators of $\Gamma(1)$ described above; I corresponds to the identity).

To get a modular form on $\Gamma(2)$ of weight -2, we will need to write g as a quotient of two modular forms of positive weight. What should the denominator be? The natural candidate, following the $+1$ case, is once again Δ. [38]

The ring of modular forms on $\Gamma(2)$ is generated by θ_{00}^4 and θ_{01}^4, each of which is of weight 2. We will want:

$$g(t) = \frac{P(\theta_{00}^4, \theta_{01}^4)}{\Delta}$$

where P is homogeneous of degree 5 (so that the numerator has weight 10, making g have weight -2 as desired). Now, we have a 6-dimensional space of modular forms of weight 10, generated by $\theta_{00}^{20-4i}\theta_{01}^{4i}$. Thus we know that:

$$g(t) = \frac{A\theta_{00}^{20} + B\theta_{00}^{16}\theta_{01}^4 + C\theta_{00}^{12}\theta_{01}^8 + D\theta_{00}^8\theta_{01}^{12} + E\theta_{00}^4\theta_{01}^{16} + F\theta_{01}^{20}}{\Delta}$$

for some still-to-be-determined constants $A, \ldots, F$.

Now, (86) implies that $g(-1/t)t^2$ necessarily expands into a series containing only odd powers of $q^{1/2}$. Substituting $-1/t$ for t and multiplying by t^2 is equivalent to replacing θ_{01} with θ_{10} and multiplying by an overall factor of -1 (since each term in the numerator is of degree 5 in θ_{00}^4 and θ_{01}^2).

$$\phi(t) = g(-1/t)t^2 = -\frac{A\theta_{00}^{20} + B\theta_{00}^{16}\theta_{10}^4 + C\theta_{00}^{12}\theta_{10}^8 + D\theta_{00}^8\theta_{10}^{12} + E\theta_{00}^4\theta_{10}^{16} + F\theta_{10}^{20}}{\Delta}$$

Now, shifting the parameter t by by 1 (i.e., $t \mapsto t+1$), we see that:

$$\phi(t+1) = \frac{-A\theta_{01}^{20} + B\theta_{01}^{16}\theta_{10}^4 - C\theta_{01}^{12}\theta_{10}^8 + D\theta_{01}^8\theta_{10}^{12} - E\theta_{01}^4\theta_{10}^{16} + F\theta_{10}^{20}}{\Delta}$$

[38] Recall that Δ is equal to the product $\frac{1}{256}\theta_{00}^8\theta_{10}^8\theta_{01}^8$.

To compare $\phi(t+1)$ and $\phi(t)$, we should get rid of all θ_{01}-terms of $\phi(t+1)$. By the Jacobi identity, we have $\theta_{01}^4 = \theta_{00}^4 - \theta_{10}^4$. Equating terms in this expansion in accordance with (86) we get:

$$\begin{aligned}\phi(t+1) =& \frac{1}{\Delta}\Big(-A\left(\theta_{00}^4-\theta_{10}^4\right)^5 + B\left(\theta_{00}^4-\theta_{10}^4\right)^4\theta_{10}^4 - C\left(\theta_{00}^4-\theta_{10}^4\right)^3\theta_{10}^8 \\ &+ D\left(\theta_{00}^4-\theta_{10}^4\right)^2\theta_{10}^{12} - E\left(\theta_{00}^4-\theta_{10}^4\right)\theta_{10}^{16} + F\theta_{10}^{20}\Big) \\ =& \frac{1}{\Delta}\Big((-A)\theta_{00}^{20} + (5A+B)\theta_{00}^{16}\theta_{10}^4 + (-10A-4B-C)\theta_{00}^{12}\theta_{10}^8 \\ &+ (10A+6B+3C+D)\theta_{00}^8\theta_{10}^{12} + (-5A-4B-3C-2D-E)\theta_{00}^4\theta_{01}^{16} \\ &+ (A+B+C+D+E+F)\theta_{10}^{20}\Big)\end{aligned}$$

which gives us:

$$\begin{aligned}A =& -A \\ B =& 5A+B \\ C =& -10A-4B-C \\ D =& 10A+6B+3C+D \\ E =& -5A-4B-3C-2D-E \\ F =& A+B+C+D+E+F\end{aligned}$$

which gives us $A=0$, $C=-2B$, $E=B-D$, with B, D and F freely specifiable.

Now we can consider the implications of (87). We note that:

$$\begin{aligned}g(-1/t)t^2 = \phi(t) =& -\frac{1}{\Delta}\Big(A\theta_{00}^{20} + B\theta_{00}^{16}\left(\theta_{00}^4-\theta_{01}^4\right) + C\theta_{00}^{12}\left(\theta_{00}^4-\theta_{01}^4\right)^2 \\ &+ D\theta_{00}^8\left(\theta_{00}^4-\theta_{01}^4\right)^3 + E\theta_{00}^4\left(\theta_{00}^4-\theta_{01}^4\right)^4 + F\left(\theta_{00}^4-\theta_{01}^4\right)^5\Big) \\ =& \frac{1}{\Delta}\Big((-A-B-C-D-E-F)\theta_{00}^{20} \\ &+ (B+2C+3D+4E+5F)\theta_{00}^{16}\theta_{01}^4 + (-C-3D-6E-10E)\theta_{00}^{12}\theta_{01}^8 \\ &+ (D+4E+10F)\theta_{00}^8\theta_{01}^{12} + (-E-5F)\theta_{00}^4\theta_{01}^{16} + (F)\theta_{01}^{20}\Big)\end{aligned}$$

and that:

$$g(t-1) = \frac{F\theta_{00}^{20} + E\theta_{00}^{16}\theta_{01}^4 + D\theta_{00}^{12}\theta_{01}^8 + C\theta_{00}^8\theta_{01}^{12} + B\theta_{00}^4\theta_{01}^{16} + A\theta_{01}^{20}}{\Delta}.$$

This leads us to the system of linear equations[39]:

[39]The parentheses are to show which terms come from each part of the sum in (87).

$$
\begin{aligned}
A &= (-A - B - C - D - E - F) + (F) \\
B &= (B + 2C + 3D + 4E + 5F) + (E) \\
C &= (-C - 3D - 6E - 10F) + (D) \\
D &= (D + 4E + 10F) + (C) \\
E &= (-E - 5F) + (B) \\
F &= (F) + (A).
\end{aligned}
$$

which gives, in combination with the previous identities, only one extra constraint: $B = 2D - 5F$. [40] Thus g is of the form:

$$
(88) \quad g(t) = \frac{(2D-5F)\theta_{00}^{16}\theta_{01}^{4} + (-4D+10F)\theta_{00}^{12}\theta_{01}^{8} + D\theta_{00}^{8}\theta_{01}^{12} + (D-5F)\theta_{00}^{4}\theta_{01}^{16} + F\theta_{01}^{20}}{\Delta}.
$$

Now, observe that

$$
\int_0^\infty e^{\pi t} e^{-\pi r^2 t} dt = \frac{1}{\pi(r^2-1)}
$$

Thus, if $g(it)$ contains an $e^{\pi t}$ term (which would correspond to a $q^{1/2}$ from the numerator multiplying the q^{-1} term coming from $1/\Delta$), then $b(r)$ will diverge at $r = \pm 1$. This is not behavior we want; so we see that the numerator has no $q^{1/2}$ term. Recall that:

$$
\begin{aligned}
\theta_{00}^{16}\theta_{01}^{4} &= 1 + 24q^{1/2} + 248q + 1376q^{3/2} + \cdots \\
\theta_{00}^{12}\theta_{01}^{8} &= 1 + 8q^{1/2} - 8q - 224q^{3/2} + \cdots \\
\theta_{00}^{8}\theta_{01}^{12} &= 1 - 8q^{1/2} - 8q + 224q^{3/2} + \cdots \\
\theta_{00}^{4}\theta_{01}^{16} &= 1 - 24q^{1/2} + 248q - 1376q^{3/2} + \cdots \\
\theta_{01}^{20} &= 1 - 40q^{1/2} + 760q - 9120q^{3/2} \cdots
\end{aligned}
$$

Setting the coefficient of $q^{1/2}$ equal to 0 in (88) gives us:

$$
\begin{aligned}
0 &= 24(2D - 5F) + 8(-4D + 10F) - 8(D) - 24(D - 5F) - 40F \\
&= 40F - 16D \\
&= 8(5F - 2D).
\end{aligned}
$$

We see that the coefficients of both $\theta_{00}^{16}\theta_{01}^{4}$ and $\theta_{00}^{12}\theta_{01}^{8}$ must vanish. Letting $K = F/2$ (so that $2K = F$ and $5K = D$), we have:

[40] It is noteworthy how much redundancy exists in these two sets of constraints. One might worry, given that there appear to be 12 such constraints, and only 6 variables, that the system is overconstrained; interestingly, there are only four independent constraints imposed on the variables $A, \ldots, F$.

$$g(t) = K\frac{5\theta_{00}^8\theta_{01}^{12} - 5\theta_{00}^4\theta_{01}^{16} + 2\theta_{01}^{20}}{\Delta}. \tag{89}$$

Now we can apply the Cohn-Miller conjectures to determine K. We recall that, since $\sin^2(\pi r^2/2) = (\pi^2/4)r^4 + O(r^8)$, the quadratic term of the Taylor expansion for

$$b(r) = \sin^2(\pi r^2/2)\int_0^\infty g(it)e^{-\pi r^2 t}dt \tag{90}$$

must come from a $1/r^2$ term in the integral. This is given by the Laplace transform of constant term of the Fourier expansion. Note that:

$$\begin{aligned} g(t) =& K\Big[5(1 - 8q^{1/2} - 8q + 224q^{3/2} + \cdots) - 5(1 - 24q^{1/2} + 248q - 1376q^{3/2} + \cdots) + 2(1 - 40q^{1/2} + 760q - \\ &\Big[q^{-1} + 24 + 324q + \cdots\Big] \\ =& K\left(2q^{-1} + 288 - 10240q^{1/2} + \cdots\right), \end{aligned}$$

so that if we want the quadratic term to be $-3/5$ as in the Cohn-Miller conjectures, we need

$$K = -\frac{1}{120\pi}.$$

Thus we let

$$g(t) = -\frac{1}{120\pi}\cdot\frac{5\theta_{00}^8\theta_{01}^{12} - 5\theta_{00}^4\theta_{01}^{16} + 2\theta_{01}^{20}}{\Delta}. \tag{91}$$

which, we expect, will give us the -1 eigenfunction $b(r)$ via (90).

4.4. **The Rigorous Argument.** Our construction of $a(r)$ and $b(r)$ almost provides us with a full proof that $a(r) + b(r)$ is the sought-after magic function. There are some details, however, which we need to address. The biggest, of course, is that we need to verify that adding $a(r)$ and $b(r)$ introduces no new sign changes. We will show this in the subsequent section. However, we must also check that the analysis we glossed over in attempting to construct these eigenfunctions is justified. In particular, we must verify the legitimacy of our contour shifting argument, we must make sure that we can swap the Fourier and Laplace transform integrals, and we need to check that our eigenfunctions behave correctly at $r = 0$ and $r = \pm\sqrt{2}$. To make certain that we have all the details in place, we shall essentially start afresh, assuming that our eigenfunctions have the form we discussed above.

This section will very much resemble Viazovska's actual paper [38], and in fact, will be logically self-contained. But to demystify the *deus ex machina* effect of Viazovska's argument, we felt it necessary to give a rather detailed explanation of how these eigenfunctions might be constructed so as to make them satisfy the desired properties before we gave the rigorous argument.

We define:

$$\phi_+(t) = \frac{\pi}{2160} \cdot \frac{(E_2E_4 - E_6)^2}{\Delta}$$

This is a holomorphic function on the upper half plane, as Δ never vanishes. It therefore has a convergent Fourier expansion:[41]

$$\begin{aligned} \phi_+ =& \pi\left(240q + 14400q^2 + 403200q^3 + 7267200q^4 + \cdots\right) \\ =& 240\pi\left(q + 60q^2 + 1680q^3 + 30280q^4 + \cdots\right). \end{aligned} \tag{92}$$

We also see from applying the modular transformation properties of E_2, E_4, and E_6, that

$$\phi_+(-1/t)t^2 = \frac{\pi}{2160} \cdot \frac{(E_2E_4 - E_6)^2}{\Delta} t^2 + \frac{i}{180} \cdot \frac{E_4E_6 - E_4^2E_2}{\Delta} t - \frac{1}{60\pi} \cdot \frac{E_4^2}{\Delta} \tag{93}$$

We call this function $g_+(t)$. We note that this automatically satisfies

$$g_+(t+1) - 2g_+(t) + g_+(t-1) = 2\phi_+(t) \tag{94}$$

since the coefficients of the powers of t are each periodic, and the second differences of 1, t, and t^2 are 0, 0, and 2, respectively. Note that the coefficients of t^2, t, and the constant term of (93) are each holomorphic quasimodular forms on $\mathbb{H}$; each is periodic. (We shall call them, as before, $\psi_1(t)$ and $\psi_2(t)$ respectively.) As with ϕ, we can expand ψ_1 and ψ_2 into convergent Fourier Series:

$$\begin{aligned} g_+(t) =& \phi(t)t^2 + \psi_1(t)t + \psi_2(t) \\ =& \pi\left(240q + 14400q^2 + 403200q^3 + 7267200q^4 + \cdots\right)t^2 \\ & + \left(-4i - 1128iq - 52320iq^2 - \cdots\right)t \\ & + \frac{1}{\pi}\left(-\frac{q^{-1}}{60} - \frac{42}{5} - \frac{6147q}{5} - \frac{134752q^2}{3} - \cdots\right) \end{aligned} \tag{95}$$

[41]This follows from elementary complex analysis. If f is a holomorphic function on $\mathbb{H}$ satisfying $f(z+1) = f(z)$, then for nonzero $z_0 \in \mathbb{D}$, the unit disk, we can take a branch of $\log(z)$ which is holomorphic on a sufficiently small neighborhood U of z_0. Note that, independent of the branch of log we choose, $\frac{1}{2\pi i}\log(z) \in \mathbb{H}$ for all $z \in U$. Thus we can take the composition $f\left(\frac{1}{2\pi i}\log(z)\right) := g(z)$, which is also holomorphic; moreover, it is independent of choice of branch of log by the periodicity of f. As g is holomorphic on $\mathbb{D}\setminus\{0\}$, it expands into a convergent Laurent series about 0; that is, $g(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ for all $z \in \mathbb{D}\setminus\{0\}$. We see that $\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z} = g\left(e^{2\pi i z}\right) = f(z)$, which gives a representation of f as a convergent q-series.

The analysis that we will perform will require us to bound the Fourier coefficients of the quasimodular form ϕ_+. Viavozska used the rather precise estimates for Fourier coefficients of negative weight modular forms that arise from the Hardy-Ramanujan circle method [33]; such precision is not actually required for our purposes.

The first thing that we notice is that (92) has only nonnegative Fourier coefficients; moreover, each q series multiplying a power of t in (95) has a consistent sign. This in itself is an interesting fact. We record it as:

Proposition 4.4.1. The coefficients of the q-expansion of the function $\phi_+(t) = \frac{\pi}{2160}\frac{(E_2E_4-E_6)^2}{\Delta}$ are all non-negative. Furthermore, the q-expansion coefficients of $i\psi_1(t)$ and $-\psi_2(t)$ are all non-negative.

Proof. The product representation for $\frac{1}{\Delta} = \frac{1}{q\prod_{n=1}^{\infty}(1-q^n)^{24}}$ yields

$$q^{-1}(1+q+q^2+\cdots)^{24}(1+q^2+q^4+\cdots)^{24}\cdots$$

which has an absolutely convergent logarithm for all $|q|<1$, with each factor converging absolutely for $|q|<1$. The expansion of this product therefore equals the q-series of Δ, which, we now see, will have all non-negative coefficients. We now claim that $(E_2E_4-E_6)^2$ has all non-negative coefficients. In fact, as given by the Ramanujan derivative identities, $E_2E_4 - E_6 = 3E_4'$, where the prime denotes the derivative $\frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$. Now, E_4 has all nonnegative q-coefficients[42] hence so does $3E_4'$, and $(3E_4')^2 = (E_2E_4-E_6)^2$. To see that $i\psi_1$ always has non-negative coefficients, note that $i\psi_1(t) = \frac{1}{180}\cdot\frac{E_4(E_2E_4-E_6)}{\Delta}$, and that $-\psi_2(t) = \frac{1}{60\pi}\cdot\frac{E_4^2}{\Delta}$ have non-negative Fourier coefficients for the same reasons.

□

Following the notation of Viazovska, let us now define $c_{\phi_+}(n)$ to be the coefficient of q^n in the expansion of ϕ_+. We will use the above proposition to give a bound on the size of $c_{\phi_+}(n)$.

Proposition 4.4.2. For $n \geq 6$,

$$c_{\phi_+}(n) \leq \frac{1}{30\pi}e^{4\pi\sqrt{n}}. \tag{96}$$

Proof. Applying Proposition 5.4.1, we can underestimate the q expansion of $\phi_+(it)$ by a single summand:

$$\phi_+(it) = \sum_{k=1}^{\infty} c_{\phi_+}(k)e^{-2\pi kt} \geq c_{\phi_+}(n)e^{-2\pi nt}$$

whence

[42] Explicitly $a_n = 240\sigma_3(n)$ for $n \geq 1$ and 1 for $n = 0$. We have also seen that this counts the number of lattice points in E_8 of magnitude $\sqrt{2n}$, which is manifestly non-negative.

$$c_{\phi_+}(n) \leq e^{2\pi nt}\phi_+(it). \tag{97}$$

We now try to pick t so as to optimize the RHS. Substituting for $1/t$ into (79), we get:

$$\phi_+(it) = \phi(i/t) - it\psi_1(i/t) - t^2\psi_2(i/t).$$

Substituting in to (97), we get:

$$\begin{aligned} c_{\phi_+}(n) \leq & e^{2\pi nt}\left(\phi(i/t) - it\psi_1(i/t) - t^2\psi_2(i/t)\right) \\ = & e^{2\pi nt}\left(\sum_{k=1}^{\infty} c_{\phi_+}(n)e^{-2\pi n/t} - it\sum_{k=0}^{\infty} c_{\psi_1}(k)e^{-2\pi k/t} - t^2\sum_{k=-1}^{\infty} c_{\psi_2}(k)e^{-2\pi k/t}\right) \\ = & e^{2\pi nt}\left(\frac{1}{60\pi}e^{2\pi/t} + \sum_{k=1}^{\infty} c_{\phi_+}(k)e^{-2\pi k/t} - it\sum_{k=0}^{\infty} c_{\psi_1}(k)e^{-2\pi k/t} - t^2\sum_{k=0}^{\infty} c_{\psi_2}(k)e^{-2\pi k/t}\right), \end{aligned}$$

where, in the last inequality, we exposed the q^{-1} term of ψ_2. Now, we let $t = \frac{1}{\sqrt{n}}$. We get:

$$\begin{aligned} c_{\phi_+}(n) \leq & e^{2\pi\sqrt{n}}\Bigg(\frac{1}{60\pi}e^{2\pi\sqrt{n}} + \sum_{k=1}^{\infty} c_{\phi_+}(k)e^{-2\pi k/\sqrt{n}} \\ & - \frac{i}{\sqrt{n}}\sum_{k=0}^{\infty} c_{\psi_1}(k)e^{-2\pi k/\sqrt{n}} - n\sum_{n=0}^{\infty} c_{\psi_2}(n)e^{-2\pi k/\sqrt{n}}\Bigg). \end{aligned}$$

For $n \geq 1$, we have, by Proposition 5.4.1:

$$\sum_{k=1}^{\infty} c_{\phi_+}(k)e^{-2\pi k/\sqrt{n}} \leq \sum_{k=1}^{\infty} c_{\phi_+}(k)e^{-2\pi k} = \phi_+(i),$$

$$-\frac{i}{\sqrt{n}}\sum_{k=0}^{\infty} c_{\psi_1}(k)e^{-2\pi k/\sqrt{n}} \leq 0,$$

$$-n\sum_{n=0}^{\infty} c_{\psi_2}(n)e^{-2\pi k/\sqrt{n}} \leq -n\sum_{n=0}^{\infty} c_{\psi_2}(n)e^{-2\pi n} = -n\psi_2(i).$$

Thus we have, for sufficiently large n:

$$c_{\phi_+}(n) \leq e^{2\pi\sqrt{n}}\left(\frac{1}{60\pi}e^{2\pi\sqrt{n}} + \phi_+(i) - n\psi_2(i)\right)$$

We can compute that $\phi_+(i) = 1.574$, and that $\psi_2(i) = -6.297$. For $n \geq 6$, we can see by elementary calculus that the LHS is strictly less than $\frac{1}{30\pi}e^{2\pi\sqrt{n}}$.

□

An entirely similar argument can be used to show:

$$|c_{\psi_1}(n)| \leq \frac{1}{30\pi} e^{4\pi\sqrt{n}} \tag{98}$$

$$|c_{\psi_2}(n)| \leq \frac{1}{30\pi} e^{4\pi\sqrt{n}}. \tag{99}$$

Definition. For $x \in \mathbb{R}^8$, we let[43]:

$$\begin{aligned} a(x) := -\frac{1}{4i}\Bigg(\int_{-1}^{i} g_+(t+1)e^{\pi i\|x\|^2 z}dt + \int_{1}^{i} g_+(t-1)e^{\pi i\|x\|^2 z}dt \\ - 2\int_{0}^{i} g_+(t)e^{\pi i\|x\|^2 z}dt + 2\int_{i}^{i\infty} \phi_+(z)\, e^{\pi i\|x\|^2 z}dt \Bigg) \end{aligned} \tag{100}$$

where the contours along which we integrate are as in Figure 6. [44]

Proposition 4.4.3. The function a defined by (100) is a Schwartz function and satisfies $\widehat{a} = a$.

Proof. Proposition 5.4.2. gives us $|c_{\phi+}(n)| \leq e^{4\pi\sqrt{n}}$ for sufficiently large n. Thus:

$$\begin{aligned} |\phi_+(z)| &\leq C\sum_{n=1}^{\infty} e^{4\pi\sqrt{n}} \left|e^{-2\pi n z}\right| & (101) \\ &= \sum_{n=1}^{\infty} e^{4\pi\sqrt{n}} \left|e^{-2\pi n \mathrm{Im}(z)}\right| \\ &\leq C' e^{-2\pi \mathrm{Im}(z)} & (102) \end{aligned}$$

when $\mathrm{Im}(z) \geq \frac{1}{2}$. Thus we can estimate, for $r \in \mathbb{R}$ with $r > 0$:

[43]We warn in advance that, like in the chapter on the Cohn-Elkies linear programming bound, we will be somewhat cavalier about the domain of this function (and similarly for the -1-eigenfunction b). Technically, $x \in \mathbb{R}^8$, and $a : \mathbb{R}^8 \to \mathbb{R}$. However, a is a radial function, depending only on $\|x\| = r$. When we write $a(r)$, we mean $a(x)$ for any x with $\|x\| = r$.

[44]Viazovska first gives this definition (although without the factor of $-1/(4i)$, which we include to make (106) have no factor of 4) in her paper, and later proves that it agrees with $\sin^2(\pi r^2/2)\int_0^{i\infty} \phi(-1/t)t^2 e^{\pi i r^2 t}$ for $r > \sqrt{2}$. While (100) is less immediately appealing, it is actually economical to define a in this way: it allows for $a(r)$ to be defined on the entire real line from the outset; the $\sin^2$-times-Laplace-transform version only converges for $r > \sqrt{2}$.

$$\left|\int_{-1}^{i}\phi_{+}\left(\frac{-1}{z+1}\right)(z+1)^2e^{\pi i\|x\|^2 z}dz\right| = \left|\int_{i\infty}^{-\frac{1}{i+1}}\phi_{+}(z)\, z^{-4}e^{\pi i r^2\left(-\frac{1}{z}-1\right)}dz\right|$$
$$\leq C\int_{\frac{1}{2}}^{\infty} e^{-2\pi t}e^{-\pi r^2/t}dt$$
$$\leq C\int_{0}^{\infty} e^{-2\pi t}e^{-\pi r^2/t}dt \tag{103}$$

Viazovska simply observes that this equals $C_2 r K_1(2\sqrt{2}\pi r)$, where K_1 is a modified Bessel function of the first kind. But we need not appeal to the theory of Bessel functions to see that this function is rapidly decreasing. Indeed, we can break the integral up[45]:

$$\int_{0}^{\infty} e^{-2\pi t}e^{-\pi r^2/t}dt = \int_{0}^{r/\sqrt{2}} e^{-\pi\left(2t+r^2/t\right)}dt + \int_{r/\sqrt{2}}^{\infty} e^{-\pi\left(2t+r^2/t\right)}dt$$

The integral on the left can be bounded above:

$$\int_{0}^{r/\sqrt{2}} e^{-\pi\left(2t+r^2/t\right)}dt \leq \int_{0}^{r/\sqrt{2}} e^{-\pi r^2/t}dt$$
$$= \int_{\sqrt{2}/r}^{\infty} e^{-\pi r^2 t}\frac{dt}{t^2}$$
$$\leq \frac{r^2}{2}\int_{\sqrt{2}/r}^{\infty} e^{-\pi r^2 t}dt$$
$$= \frac{r^2}{2}\cdot\frac{e^{-\sqrt{2}\pi r}}{\pi r^2}$$
$$= \frac{e^{-\sqrt{2}\pi r}}{2\pi},$$

and the second integral can be bounded:

$$\int_{r/\sqrt{2}}^{\infty} e^{-\pi\left(2t+r^2/t\right)}dt \leq \int_{r/\sqrt{2}}^{\infty} e^{-2\pi t}dt$$
$$= \frac{e^{-\sqrt{2}\pi r}}{2\pi}.$$

Thus (103) is always bounded by $\frac{e^{-\sqrt{2}\pi r}}{\pi}$, whence this summand of $a(r)$ decays faster than any inverse power of r. By identical reasoning, this shows that the second and third integrals in (100) also decay faster than any inverse power $\|x\|^\alpha$.

For the final integral in (100), note that:

[45] The point $t = r/\sqrt{2}$ is so chosen as the be the minimum of the function $2t + r^2/t$.

$$\left|\int_{i}^{i\infty}\phi_{+}(z)e^{\pi i r^2 z}dz\right| \leq C\int_{1}^{\infty}e^{-2\pi t}e^{-\pi r^2 t}dt = C'\frac{e^{-\pi\left(r^2+2\right)}}{r^2+2}$$

We see, therefore, that all four of the summands for $a(x)$ decay faster than any inverse power of r.

Now, say we apply some derivative d/dx^{α} to $a(x)$ (with α an 8-dimensional multi-index). We will replace the Gaussian factor in each integrand by some polynomial $p(z, x_1, \ldots, x_8)$ in the components x_i and z, multiplied by the Gaussian $e^{-\pi\|x\|^2 t}dt$. The above estimates will become:

$$\left|\int_{-1}^{i}\phi_{+}\left(\frac{-1}{z+1}\right)(z+1)^2 p_{\alpha}(z, x_1, \ldots x_8)e^{\pi i\|x\|^2 z}dz\right| \tag{104}$$

and

$$\left|\int_{i}^{i\infty}\phi_{+}(z)e^{\pi i\|x\|^2 z}p_{\alpha}(z, x_1, \ldots x_8)dz\right| \tag{105}$$

For (104), we can let $C(x_1 \ldots x_8)$ denote the maximum of $|p(z, x_1, \ldots, x_8)|$ on the circular arc from $z=-1$ to $z=i$. As this arc is compact, this maximum exists; moreover, $C(x_1, \ldots x_8)$ is bounded by a polynomial in $x_1, \ldots, x_8$. We know already that $\left|\int_{-1}^{i}\phi_{+}\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i\|x\|^2 z}dz\right|$ decays faster than any inverse power of r, hence $C(x_1, \cdots, x_8)\left|\int_{-1}^{i}\phi_{+}\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i\|x\|^2 z}dz\right|$ is bounded. The same estimate applies to the derivative d/dx^{α} of the 1-to-i and 0-to-i contour integrals. For (105), the computation reduces to the fact that a multidimensional Gaussian lies in Schwartz space.[46] We can thus conclude $a(x)$ is a Schwartz function.

We shall now check that $\widehat{a}=a$. Note that, writing everything in terms of ϕ_{+}, the Laplace transform $\widehat{a}(y)$ is given by:

$$\begin{aligned}
-4i\int_{\mathbb{R}^8}a(x)e^{2\pi i\langle x,y\rangle}dy = \int_{\mathbb{R}^8}\Bigg(&\int_{-1}^{i}\phi_{+}\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i\|x\|^2 z}dz \\
&+\int_{1}^{i}\phi_{+}\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i\|x\|^2 z}dz \\
&-2\int_{0}^{i}\phi_{+}\left(\frac{-1}{z}\right)z^2 e^{\pi i\|x\|^2 z}dz \\
&+2\int_{i}^{i\infty}\phi_{+}(z)\, e^{\pi i\|x\|^2 z}dz\Bigg)e^{2\pi i\langle x,y\rangle}dy
\end{aligned}$$

[46]See [23].

The estimates discussed above suffice to demonstrate that each of the four double integrals converges absolutely. Hence we are justified in swapping the order of integration. The Fourier transform of an 8-dimensional Gaussian is given by:

$$\mathcal{F}\left(e^{\pi i\|x\|^2 z}\right)(y) = z^{-4} e^{\pi i \|y\|^2\left(\frac{-1}{z}\right)}.$$

We thus obtain:

$$\begin{aligned}
-4i\widehat{a}(x) = &\int_{-1}^{i} \phi_+\left(\frac{-1}{z+1}\right)(z+1)^2 z^{-4} e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)} dz \\
&+ \int_{1}^{i} \phi_+\left(\frac{-1}{z-1}\right)(z-1)^2 z^{-4} e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)} dz \\
&- 2\int_{0}^{i} \phi_+\left(\frac{-1}{z}\right) z^2 z^{-4} e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)} dz \\
&+ 2\int_{i}^{i\infty} \phi_+(z)\, z^{-4} e^{\pi i\|x\|^2\left(\frac{-1}{z}\right)} dz
\end{aligned}$$

We now substitute $t = -1/z$, and verify that the first two summands get swapped and the last two summands get swapped. We shall see that our argument uses only the periodicity of ϕ_+. Indeed:

$$\begin{aligned}
-4i\widehat{a}(x) = &\int_{1}^{i} \phi_+\left(1 - \frac{1}{t-1}\right)\left(\frac{-1}{t}+1\right)^2 t^2 e^{\pi i\|x\|^2 t} dt \\
&+ \int_{-1}^{i} \phi_+\left(1 - \frac{1}{t+1}\right)\left(\frac{-1}{t}-1\right)^2 t^2 e^{\pi i\|x\|^2 t} dt \\
&- 2\int_{i\infty}^{0} \phi_+(t)\, e^{\pi i\|x\|^2 t} dt \\
&+ 2\int_{i}^{0} \phi_+\left(-\frac{1}{t}\right) t^2 e^{\pi i\|x\|^2 t} dt \\
= &\int_{1}^{i} \phi_+\left(-\frac{1}{t-1}\right)(t-1)^2 e^{\pi i\|x\|^2 t} dt + \int_{-1}^{i} \phi_+\left(-\frac{1}{t+1}\right)(t+1)^2 e^{\pi i\|x\|^2 t} dt \\
&+ 2\int_{0}^{i\infty} \phi_+(t)\, e^{\pi i\|x\|^2 t} dt - 2\int_{0}^{i} \phi_+\left(-\frac{1}{t}\right) t^2 e^{\pi i\|x\|^2 t} dt \\
= &-4ia(x)
\end{aligned}$$

whence $\widehat{a}(x) = a(x)$. This completes the proof of Proposition 5.4.2.

□

Now we check that our global definition of $a(r)$ agrees with the $\sin^2$-times-Laplace-transform version. Indeed:

Proposition 4.4.4. For $r > \sqrt{2}$, we have:

$$a(r) = \sin^2(\pi r^2/2) \int_0^\infty g_+(t) e^{-\pi r^2 t} dt. \tag{106}$$

Proof. Following Viazovska, we let the right hand side be $d(r)$. From (95), we see that:

$$\begin{aligned}
\phi_+\left(\frac{-1}{it}\right) =& O\left(e^{-2\pi/t}\right) \text{ as } t \to 0,\\
\phi_+\left(\frac{-1}{it}\right) =& O, \left(t^2 e^{2\pi t}\right) \text{ as } t \to \infty
\end{aligned}$$

which shows that (106) converges absolutely. Now we expand $\sin^2(\pi r^2/2)$ as $-\frac{1}{4}\left(q - 2 + q^{-1}\right)$. So we write:

$$\begin{aligned}
-4id(r) = &\int_{-1}^{-1+i\infty} g_+(t+1) e^{\pi i r^2 t} dt - 2\int_0^{i\infty} g_+(t) e^{\pi i r^2 t} dt \\
&+ \int_1^{1+i\infty} g_+(t-1) e^{\pi i r^2 t} dt.
\end{aligned}$$

Now, we want to be able to shift the contours to those of Figure 6, so that we can apply (94). We have already observed that g_+ is holomorphic in $\mathbb{H}$. Moreover, when $r > \sqrt{2}$, then, as $\mathrm{Im}(t) \to \infty$,

$$\left| g_+(t) e^{\pi i r^2 t} \right| \to 0,$$

since, as we see from (95), $g_+(t)$ is dominated by the $q^{-1} = e^{-2\pi i t}$ term, which is killed by the $e^{\pi i r^2 t}$-term when $r > \sqrt{2}$. Thus, for $r > \sqrt{2}$, we can deform the contours to those of Figure 6, giving:

$$\begin{aligned}
-4id(r) &= \int_i^{i\infty} \left(g_+(t+1) - 2g_+(t) + g_+(t-1)\right) e^{i\pi r^2 t} dt + \int_i^0 2g_+(t) e^{i\pi r^2 t} dt \\
&+ \int_{-1}^i g_+(t+1) e^{i\pi r^2 t} dt + \int_1^i g_+(t-1) e^{i\pi r^2 t} dt. \\
&= 2\int_i^{i\infty} \phi_+(t) e^{i\pi r^2 t} dt - 2\int_o^i g_+(t) e^{i\pi r^2 t} dt \\
&+ \int_{-1}^i g_+(t+1) e^{i\pi r^2 t} dt + \int_1^i g_+(t-1) e^{i\pi r^2 t} dt. \\
&= -4ia(r)
\end{aligned}$$

whence $d(r) = a(r)$ for $r > \sqrt{2}$, finishing the proof.

□

We now wish to verify the correct vanishing behavior of $a(r)$ around $r = 0$ and $r = \pm\sqrt{2}$. Recall that the we normalized g_+ so that the principal terms of the q-expansion would have Laplace transforms with the correct pole behavior at $r = 0$ and $r = \pm\sqrt{2}$. We will remove these terms from (106). Note that the term in the expansion (95) of $g_+(t)$ that forces us to take $r > \sqrt{2}$ is the $e^{2\pi t}$-term coming from the q^{-1}. If we remove this term, our domain for (106) can be extended.

Proposition 4.4.5. For all $r \in \mathbb{R}$, we have:

$$a(r) = \sin^2(\pi r^2/2)\Bigg(\frac{1}{4\pi^2 r^4} - \frac{42}{5\pi^2 r^2} - \frac{1}{60\pi^2\,(r^2-2)} + \int_0^\infty \left(g_+(it) - 4t + \frac{42}{5\pi} + \frac{e^{2\pi t}}{60\pi}\right) e^{-\pi r^2 t} dt\Bigg). \tag{107}$$

Proof. We note from (95) that $g_+(it) = 4t - \frac{e^{2\pi t}}{60\pi} - \frac{42}{5\pi} + O\left(t^2 e^{-2\pi t}\right)$ as $t \to \infty$. Thus the integral in (107) converges absolutely for all r. When $r > \sqrt{2}$, we see that:

$$\int_0^\infty \left(4t - \frac{42}{5\pi} - \frac{e^{2\pi t}}{60\pi}\right) e^{-\pi r^2 t} dt = \frac{4}{\pi^2 r^4} - \frac{42}{5\pi^2 r^2} - \frac{1}{60\pi^2(r^2-2)}$$

so that (107) holds when $r > \sqrt{2}$. However, due the the absolute convergence of the integral in (107), we see that the RHS extends to a holomorphic function on a neighborhood of the real line. But $a(r)$, as defined by (100), is also a holomorphic function on a neighborhood of the real line. These two functions agree for all $r > \sqrt{2}$, thus (107) must be true for all r. □

Corollary. We have :

$$a(0) = 1, \qquad a(\sqrt{2}) = 0, \qquad a'(\sqrt{2}) = -\frac{1}{60\sqrt{2}},$$

and:

$$a(r) = 1 - \frac{21}{10} r^2 + O(r^4).$$

Proof. Plug in to Proposition 5.4.4, noticing that the only poles which will cancel the double-zeroes of $\sin^2(\pi r^2/2)$ come from the terms outside the integral. The Taylor series comes from expanding $\sin^2(\pi r^2/2)$, multiplying by the $1/r^2$ and $1/r^4$ terms, and noticing that the other terms multiplying $\sin^2(\pi r^2/2)$ in (107) are all holomorphic at 0. (Alternatively, note that the function $a(r)$ is even.)

□

We now turn to the analysis of the -1 eigenfunction. The -1 eigenfunction is $\sin^2$ times the Laplace transform of a bona fide modular form (as opposed to a quasimodular form, as in the $+1$ case), so we can immediately define $g_-(t)$ without first defining $\phi_-(t) = g_-(-1/t)t^2$. Let:

$$g_-(t) := -\frac{1}{120\pi} \cdot \frac{5\theta_{00}^8\theta_{01}^{12} - 5\theta_{00}^4\theta_{01}^{16} + 2\theta_{01}^{20}}{\Delta}. \tag{108}$$

This defines a holomorphic function on $\mathbb{H}$, which is periodic of period 2, and modular of weight -2 on $\Gamma(2)$. Thus we can expand as a convergent $q^{1/2}$-series:

$$g_-(t) = -\frac{1}{60\pi}q^{-1} - \frac{12}{5\pi} + \frac{256}{3\pi}q^{1/2} - \frac{5877}{5\pi}q + \frac{52224}{5\pi}q^{3/2} - \frac{213280}{3\pi}q^2 + O\left(q^{5/2}\right). \tag{109}$$

By construction, we know that $g_-(t)$ satisfies

$$g_-(-1/t)t^2 + g_-(t-1) = g_-(t), \tag{110}$$

and that $\phi_-(t) := g(-1/t)t^2$ satisfies

$$\phi_-(t+1) = -\phi_-(t). \tag{111}$$

As we have already mentioned, Viazovska calls the three summands of (109) (up to a constant factor) by ψ_S, ψ_T and ψ_I, respectively. We give their Fourier expansions, too:

$$\begin{aligned} g_-(t+1) = g_-(t-1) = &- \frac{1}{60\pi}q^{-1} - \frac{12}{5\pi} - \frac{256}{3\pi}q^{1/2} \\ &- \frac{5877}{5\pi}q - \frac{52224}{5\pi}q^{3/2} - \frac{213280}{3\pi}q^2 \\ &+ O\left(q^{5/2}\right) \end{aligned} \tag{112}$$

$$\begin{aligned} g_-(-1/t)t^2 = \phi_-(t) = &\frac{512}{3\pi}q^{1/2} + \frac{104448}{5\pi}q^{3/2} + \frac{4027392}{5\pi}q^{5/2} \\ &+ \frac{88256512}{5\pi}q^{7/2} \\ &+ O\left(q^{9/2}\right) \end{aligned} \tag{113}$$

We note that both $g_-(t+1)$ and $g_-(-1/t)t^2$ have consistently non-positive, and consistently non-negative Fourier coefficients, respectively. We record this as:

Proposition 4.4.6. The $q^{1/2}$-series for $g(t-1)$ has coefficients which are all non-positive. The coefficients of the $q^{1/2}$-series for $g(-1/t)t^2$ are each non-negative, with the coefficient of each integral power of q (even power of $q^{1/2}$) equaling 0. Moreover, the coefficients of $q^{k/2}$ for k odd are in a ratio of $1 : -1 : 2$ in $g_-(t)$, $g_-(t+1)$, and $g_-(-1/t)t^2$, respectively.

Proof. Observe that the vanishing of the even powers of $q^{1/2}$ in $g_-(-1/t)t^2$ follows from (111). Next observe that $g_-(t)+g_-(t+1)$ is periodic of period 1, and therefore has no $q^{k/2}$ terms for odd k. Hence we see that $g_-(-1/t)t^2+2g_-(t+1)=g_-(t)+g_-(t-1)$ by (110), which, similarly, cannot have any $q^{k/2}$ terms for odd k. This demonstrates that the coefficients of $q^{k/2}$ for k odd are in a ratio of $1:-1:2$ in $g_-(t)$, $g_-(t+1)$, and $g_-(-1/t)t^2$, respectively.

Now, observe that

$$g_-(t-1) = -\frac{1}{120\pi}\cdot\frac{5\theta_{10}^8\theta_{00}^{12}-5\theta_{10}^4\theta_{00}^{16}+2\theta_{00}^{20}}{\Delta}.$$

The negativity of the coefficients of $g_-(t-1)$ follows from the fact that $1/\Delta$ has an all-positive Fourier expansion, and since the numerator of $g_-(t)$, that is, $5\theta_{10}^8\theta_{00}^{12}-5\theta_{10}^4\theta_{00}^{16}+2\theta_{00}^{20}=2(1+120q+5120q^{3/2}+67320q^2+503808q^{5/2}+2607840q^3+\cdots)$, is equal to $(2\Theta(t)+\Theta(t+1))/3$, where $\Theta(t)$ is the theta series of the lattice "DualExtremal(20,2)a".[47] This identity can be proved by the usual procedure of verifying $\Gamma(2)$-modularity of $(2\Theta(t)+\Theta(t+1))/3$ (which, as always, results from a simple application of Poisson summation) and checking that the two q-expansions agree for a finite number (in this case, $\dim M_{10}(\Gamma(2))=6$) of q-coefficients. The expansion of $\Theta(t)$ and thus of $5\theta_{10}^8\theta_{00}^{12}-5\theta_{10}^4\theta_{00}^{16}+2\theta_{00}^{20}$ manifestly has all nonnegative coefficients; hence $g_-(t-1)$ has all negative coefficients. The sign of $g(-1/t)t^2$ follows from the properties of the odd powers of $q^{1/2}$ in g_-, $g_-(t-1)$ and $g_-(-1/t)t^2$ that we demonstrated previously.

As before, we write the coefficient of q^n in the q-expansion of $f(t)$ as $c_f(n)$ for periodic functions f. Note that for $f=g_-(t)$, $g_-(t-1)$, and $g_-(-1/t)t^2$, we can have n be a half-integer.

Proposition 4.4.7. We have, for $n\geq 6$, $n\in\frac{1}{2}\mathbb{Z}_{\geq 0}$:

$$|c_{g_-}(n)| \leq \frac{1}{30\pi}e^{4\pi\sqrt{n}}, \tag{114}$$

$$|c_{g_-(-1/t)t^2}(n)| \leq \frac{1}{30\pi}e^{4\pi\sqrt{n}}, \tag{115}$$

$$|c_{g_-(t+1)}(n)| \leq \frac{1}{30\pi}e^{4\pi\sqrt{n}}. \tag{116}$$

Proof. The proof of the inequalities for the coefficients is now essentially the same as the proof of Proposition 5.4.2. We start with the estimates for $g_-(t-1)$ and $g_-(-1/t)t^2$. For each of these, we apply Proposition 5.4.6 to bound the series by a single term, apply the modular transformation property, and then optimize the value t (letting $t=1/\sqrt{n}$, as before). This gives us the above bounds. □

As in (102), we can use this estimate to immediately give us:

[47] Details of this lattice can be found in the online "Catalogue of Lattices" at http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/DualExtremal_20_2a.html

$$\left|g_-(-1/z)z^2\right| < Ce^{-\pi \mathrm{Im}(z)} \tag{117}$$

when $\mathrm{Im}(z) > \frac{1}{2}$.

Definition. Let $x \in \mathbb{R}^8$. We now define:

$$b(x) := -\frac{1}{4i}\Bigg(\int_{-1}^{i} g_-(z+1)e^{-\pi i\|x\|^2 z}dz + \int_{1}^{i} g_-(z-1)e^{-\pi i\|x\|^2 z}dz \\ - 2\int_0^i g_-(z)e^{-\pi i\|x\|^2 z}dz - 2\int_i^{i\infty} g_-\left(-\frac{1}{z}\right) z^2 e^{-\pi i\|x\|^2 z}dz\Bigg) \tag{118}$$

where the contours are as in Figure 6.

Proposition 4.4.8. The function b is a Schwartz function and satisfies

$$\widehat{b}(x) = -b(x).$$

Proof. First we shall establish that b is in Schwartz space. We have:

$$\begin{aligned}\int_{-1}^{i} g_-(z-1)e^{-\pi i\|x\|^2 z}dz &= \int_0^{i+1} g_-(z-1)e^{-\pi i\|x\|^2 z}dz \\ &= \int_{i\infty}^{-1/(i+1)} g_-\left(-\frac{1}{z}\right) e^{\pi i r^2(-\frac{1}{z}-1)}z^{-2}dz\end{aligned}$$

Recall from (117) that:

$$\left|g_-(-1/z)z^2\right| < Ce^{-\pi \mathrm{Im}(z)}$$

when $\mathrm{Im}(z) > \frac{1}{2}$. The same estimates as in Proposition 5.4.3. now apply.

To see that b defines a -1-eigenfunction, we note that absolute convergence of the double integral given by the Fourier transform and the Laplace transform permits us two swap the two integrals as we did in Proposition 5.4.2. This gives us:

$$\begin{aligned}-4i\mathcal{F}(b)(x) = &\int_{-1}^{i} g_-(z+1)z^{-4}e^{\pi i\|x\|^2(-\frac{1}{z})}dz + \int_{1}^{i} g_-(z-1)z^{-4}e^{\pi i\|x\|^2(-\frac{1}{z})}dz \\ &- 2\int_0^i g_-(z)z^{-4}e^{\pi i\|x\|^2(-\frac{1}{z})}dz - 2\int_i^{i\infty} g_-\left(-\frac{1}{z}\right) z^2 z^{-4}e^{\pi i\|x\|^2(-\frac{1}{z})}dz\end{aligned}$$

We make our perennial substitution $z \mapsto -1/z$, giving:

$$
\begin{aligned}
-4i\mathcal{F}(b)(x) = &\int_1^i g_-\left(-\frac{1}{z}+1\right) z^2 e^{\pi i\|x\|^2 z} dz + \int_{-1}^i g_-\left(-\frac{1}{z}-1\right) z^2 e^{\pi i\|x\|^2 z} dz \\
&- 2\int_{\infty i}^0 g_-\left(-\frac{1}{z}\right) z^2 e^{\pi i\|x\|^2 z} dz - 2\int_i^0 g_-\left(z\right) z^2 e^{\pi i\|x\|^2 z} dz
\end{aligned}
$$

Recal that $\phi_-(t) = g_-(-1/t)t^2$ satisfies (111), which, as we have seen, is equivalent to g_- satisfying (84) and (85). So the above is equal to:

$$
\begin{aligned}
-4i\widehat{b}(x) = &- \int_{-1}^i g_-(z+1) e^{-\pi i\|x\|^2 z} dz - \int_1^i g_-(z-1) e^{-\pi i\|x\|^2 z} dz \\
&+ 2\int_0^i g_-(z) e^{-\pi i\|x\|^2 z} dz + 2\int_i^{i\infty} g_-\left(-\frac{1}{z}\right) z^2 e^{-\pi i\|x\|^2 z} dz \\
= &- 4ib(x)
\end{aligned}
$$

completing the argument.

□

Proposition 4.4.9. For $r > \sqrt{2}$, the function $b(r)$ as defined in (118) satisfies:

$$
b(r) = \sin^2(\pi r^2/2) \int_0^{i\infty} g_-(t) e^{-\pi i r^2 t} dt \tag{119}
$$

Proof. Following Viazovska, we define the RHS of (119) to equal some function $c(r)$. We note that:

$$
\begin{aligned}
g_-(it) = &O\left(t^2 e^{-\pi/t}\right) \text{ as } t \to 0, \\
g_-(it) = &O\left(e^{2\pi t}\right) \quad \text{ as } t \to \infty.
\end{aligned}
$$

Thus we can deform the path of integration into that of Figure 6. We get:

$$
\begin{aligned}
c(r) = &\int_{-1}^i g_-(z+1) e^{\pi i r^2 t} dz + \int_1^i g_-(z-1) e^{\pi i r^2 t} dz \\
&- 2\int_0^{\infty} g_-(t) e^{\pi i r^2 t} dt + 2\int_i^{i\infty} \left(g_-(t-1) - g(t)\right) e^{\pi i r^2 t} dt \\
= &\int_{-1}^i g_-(z+1) e^{\pi i r^2 t} dz + \int_1^i g_-(z-1) e^{\pi i r^2 t} dz \\
&- 2\int_0^{\infty} g_-(t) e^{\pi i r^2 t} dt - 2\int_i^{i\infty} g\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 t} dt \\
= &b(r)
\end{aligned}
$$

This completes the proof.

□

Proposition 4.4.10. For all $r \in \mathbb{R}$, we have:

$$b(r) = \sin^2(\pi r^2/2)\left(-\frac{12}{5\pi^2 r^2} - \frac{1}{60\pi^2(r^2-2)} + \int_0^\infty \left(g_-(it) + \frac{12}{5\pi} + \frac{e^{2\pi t}}{60\pi}\right)e^{-\pi r^2 t}dt\right) \tag{120}$$

with the integral converging absolutely for all $r \in \mathbb{R}$.

Proof. This is analogous to Proposition 5.4.5. Say $r > \sqrt{2}$. Then Proposition 5.4.9 implies that

$$b(r) = \sin^2(\pi r^2/2)\int_0^\infty g_-(it)e^{-\pi r^2 t}dt$$

and by (109) see:

$$g_-(t) = -\frac{12}{5\pi} - \frac{e^{2\pi t}}{60\pi} + O\left(e^{-2\pi t}\right)$$

as $t \to \infty$. For $r > \sqrt{2}$, we have:

$$\int_0^\infty \left(-\frac{12}{5\pi} - \frac{e^{2\pi t}}{60\pi}\right)e^{-\pi r^2 t}dt = -\frac{12}{5\pi^2 r^2} - \frac{1}{60\pi^2(r^2-2)}$$

whence (120) holds for all $r > \sqrt{2}$.

But the RHS of (120) extends to a holomorphic function in a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ (the poles in the second factor are canceled by the double-zeroes of the $\sin^2$ factor). Likewise, $b(r)$, as defined in (118), extends to a holomorphic function in a neighborhood in $\mathbb{C}$ of the real line $\mathbb{R}$. These two agree for all $r > \sqrt{2}$. Thus (120) holds on all of $\mathbb{R}$.

□

Corollary. We have:

$$b(0) = 0 \qquad b(\sqrt{2}) = 0 \qquad b'(\sqrt{2}) = -\frac{1}{60\sqrt{2}}$$

and

$$b(r) = -\frac{3}{5}r^2 + O(r^4)$$

Proof. Plug the above values into (120). To see the Taylor series, expand the Taylor series of $\sin^2(\pi r^2/2)$ around $r = 0$ and expand (120). One can see that no odd powers will appear as (120) only depends upon r^2 and so is even.

□

Note that, if we let our magic function $f(r)$ equal $a(r)+b(r)$, the two corollaries together imply the Cohn-Miller conjectures and show that $f'(\sqrt{2}) = a'(\sqrt{2}) + b'(\sqrt{2}) = -\frac{1}{30\sqrt{2}} < 0$ while $\widehat{f}'(\sqrt{2}) = a'(\sqrt{2}) - b'(\sqrt{2}) = 0$, exactly as needed. The final step in the argument is to verify that no new vanishing occurs when we take the sum $f(r) = a(r) + b(r)$, which we shall attend to in the next section.

4.5. **Completing the Proof: An Inequality of Modular Forms.** We have:

Theorem 4.5.1. Let $x \in \mathbb{R}^8$, and let $a(x)$ and $b(x)$ be defined as in (100) and (118) respectively. Then let

$$f(x) = a(x) + b(x)$$

Then $f : \mathbb{R}^8 \to \mathbb{R}$ is a is a radial Schwartz function which vanishes at all non-zero magnitudes of the E_8 lattice. (As always, we somewhat abuse notation and write $f(r)$ when we mean $f(x)$ for some x such that $\|x\| = r$.) That is, $f(x) = 0$ for all x such that $\|x\| = \{\sqrt{2k}\}$, for $k \in \mathbb{Z}$, $k \geq 1$. Moreover, f satisfies:

1) For all x such that $\|x\| > \sqrt{2}$, $f(x) \leq 0$.

2) For all x, $\widehat{f}(x) > 0$.

3) $f(0) = \widehat{f}(0) = 1$.

Thus f is an optimal function in 8 dimensions for the Cohn-Elkies linear programming bound. This immediately implies that the E_8 lattice packing is the densest sphere packing in dimension 8.

Proof. We note that, by the eigenfunction property of a and b, $f(x) = a(x) + b(x)$ and $\widehat{f}(x) = a(x) - b(x)$. The $\sin^2$ factor in propositions 5.4.4 and 5.4.9 showed that a and b both vanish to order 2 at $r = \sqrt{k}$, k an even positive integer greater than or equal to 4. Furthermore, the two corollaries of the previous section showed that $f(0) = \widehat{f}(0) = 1$, which demonstrates 3). Moreover, both a and b were proved to be Schwartz functions, and both were by construction radial. Therefore f, their sum, is too. When $r = \sqrt{2}$, $f(r) = 0$ and $f'(r) < 0$, while $\widehat{f}(r) = 0$. But we must verify that no unexpected sign changes occur; i.e., that no other zeroes are introduced when we take the sum $a(x) + b(x)$.

Proposition 4.5.2. Let $A(t) = g_+(t) + g_-(t)$. Then $A(t) < 0$ for all $t \in (0, \infty)$.

Note that this Proposition states something stronger than the inequality $a(x)+b(x) \leq 0$. It states that the integrand in (106) is pointwise negative along the imaginary axis.

Proof. By (79) we can write:

$$A(t) = -t^2\phi_+(i/t) - t^2\phi_-(i/t)$$

and we can also write:

$$A(t) = -t^2\phi_+(it) + it\psi_1(it) + \psi_2(it) + g_-(it)$$

The first of these gives us an expansion in terms of $e^{-\pi/t}$; the latter gives us an expansion in terms of $e^{-\pi t}$. For $n \geq 0$ a positive integer, we let $A_0^{(n)}$ and $A_\infty^{(n)}$ be the partial q-expansions of $A(t)$ for which:

$$A(t) = A_0^{(n)}(t) + O\left(t^2 e^{-\pi n/t}\right) \quad \text{as} \quad t \to 0 \tag{121}$$

$$A(t) = A_\infty^{(n)}(t) + O\left(t^2 e^{-\pi n t}\right) \quad \text{as} \quad t \to \infty, \tag{122}$$

For example, if $n = 6$, (95) and (109) give us:

$$\begin{aligned} A_\infty^{(6)} = &- \frac{1}{30\pi}e^{2\pi t} - \frac{54}{5\pi} + \frac{256}{3\pi}e^{-\pi t} - \frac{12024}{5\pi}e^{-2\pi t} + \frac{52224}{5\pi}e^{-3\pi t} \\ &- \frac{348032}{3\pi}e^{-4\pi t} + \frac{2013696}{5\pi}e^{-5\pi t} \\ &+ t\left(4 + 1128e^{-2\pi t} + 52320e^{-4\pi t}\right) \\ &- t^2\pi(240e^{-2\pi t} + 14400e^{-4\pi t}). \end{aligned}$$

Similarly, we have:

$$\begin{aligned} A_0^{(6)} = &t^2\Big(- \frac{512}{3\pi}e^{-\pi/t} - 240\pi e^{-2\pi/t} - \frac{104448}{5\pi}e^{-3\pi/t} \\ &- 14400^{-4\pi/t} - \frac{4027392}{5\pi}e^{-5\pi/t}\Big). \end{aligned}$$

Note that $A_\infty^{(n)}$ is dominated by $-\frac{1}{30\pi}e^{2\pi t}$ as $t \to \infty$. So we expect that for sufficiently large t, $A_\infty^{(n)}$ will be negative. Likewise, note that $A_0^{(n)}$ is dominated by $-\frac{512}{3\pi}e^{-\pi/t}$. So we have $A_\infty^{(n)}(t) < 0$ as $t \to 0$. We must verify that there is no "moderately sized" value of t for which $A(t) = 0$. So we shall pick an n so that both $A_\infty^{(n)}$ and $A_0^{(n)}$ are sufficiently close to $A(t)$, and then use the fact that both are finite polynomials in $e^{-\pi t}$ and $e^{-\pi/t}$ to show that there are no other zeroes. It turns out to suffice to take $n = 6$.

Moreover, we can apply the bounds (96), (98), (99), (114), and (115) to yield:

$$\left|A(t) - A_0^{(m)}(t)\right| \leq \frac{1}{30\pi}\left(t^2 + 1\right)\sum_{n=m}^{\infty} e^{2\sqrt{2}\pi\sqrt{n}}e^{-\pi n/t} \tag{123}$$

and

$$\left|A(t) - A_\infty^{(m)}(t)\right| \leq \frac{1}{30\pi}\left(t^2 + t + 1\right)\sum_{n=m}^{\infty} e^{2\sqrt{2}\pi\sqrt{n}}e^{-\pi n t} \tag{124}$$

Let us denote the RHS of (123) and (124) by $R_0^{(m)}$ and $R_0^{(\infty)}$, respectively. We can now determine the maximum value of $A_0^{(6)}$ and $A_\infty^{(6)}$ on the intervals $(0,1]$ and $[1,\infty)$. (Note that this means the *minimum* of the absolute values, $|A_0^{(6)}|$ and $|A_\infty^{(6)}|$.) We verify that:

$$\begin{aligned}
&\left|R_0^{(6)}\right| \leq \left|A_0^{(6)}\right| \text{ for } \quad t \in (0,1] \\
&\left|R_\infty^{(6)}\right| \leq \left|A_\infty^{(6)}\right| \text{ for } t \in [1,\infty) \\
&A_0^{(6)}(t) < 0 \text{ for } t \in (0,1] \\
&A_\infty^{(6)}(t) < 0 \text{ for } t \in [1,\infty).
\end{aligned}$$

This shows us that $A(t) < 0$ for all $t \in (0,\infty)$, and therefore that f satisfies 1).

□

Proposition 4.5.3. Let $B(t) = g_+(t) - g_-(t)$. Then $B(t) > 0$ for all t.

Once again, observe this means that the integrand in the definition of our magic function $\widehat{f}$ is pointwise negative. This is somewhat stronger than the inequality in 2).

Proof. We know that $\widehat{f}(x) = a(x) - b(x)$. By Propositions 5.4.4 and 5.4.9., we know that for $r > \sqrt{2}$,

$$\widehat{f}(r) = \sin^2(\pi r^2/2) \int_0^\infty (g_+(t) - g_-(t)) e^{-\pi r^2 t} dt \tag{125}$$

However, note that the q^{-1} terms in $g_+(t)$ and $g_-(t)$ – both of which are $-\frac{q^{-1}}{60\pi}$ – cancel, whence we see that the integrand in (125) actually converges absolutely for all $r > 0$. The integral therefore extends to a holomorphic function on a complex neighborhood of $\mathbb{R} \setminus \{0\}$ which agrees with $\widehat{f}(r)$ for all $r > \sqrt{2}$. Thus $\widehat{f}$ must equal the RHS of (125) for all $r \in \mathbb{R}\setminus\{0\}$.

Following Viazovska, we let $B(t) = g_+(t) - g_-(t)$. We now proceed as in Proposition 5.5.1. We have:

$$\begin{aligned}
B(t) &= - t^2 \phi_+(i/t) + \phi_-(i/t) t^2 \\
B(t) &= - t^2 \phi_+(it) + it\psi_1(it) + \psi_2(it) - g_-(it)
\end{aligned}$$

As before, we define $B_0^{(n)}(t)$ and $B_\infty^{(n)}(t)$ to be the partial Fourier expansions of the above satisfying:

$$\begin{aligned}
B(t) =& B_0^{(n)}(t) + O\left(t^2 e^{-\pi n/t}\right) \text{ as } t \to 0 \\
B(t) =& B_\infty^{(n)}(t) + O\left(t^2 e^{-\pi n t}\right) \text{ as } t \to \infty
\end{aligned}$$

We can choose $n = 6$:

$$B_0^{(6)} = t^2\Big(\frac{512}{3\pi}e^{-\pi/t} - 240\pi e^{-2\pi/t} + \frac{104448}{5\pi}e^{-3\pi/t} - 14400^{-4\pi/t} + \frac{4027392}{5\pi}e^{-5\pi/t}\Big).$$

and

$$\begin{aligned}B_\infty^{(6)}(t) = &-\frac{6}{\pi} - \frac{256}{3\pi}e^{-\pi t} - \frac{54}{\pi}e^{-2\pi t} - \frac{52224}{5\pi}e^{-3\pi t} \\ &+ \frac{26176}{\pi}e^{-4\pi t} - \frac{2013696}{5\pi}e^{-5\pi t} \\ &+ t\left(4 + 1128e^{-2\pi t} + 52320e^{-4\pi t}\right) \\ &- t^2\pi(240e^{-2\pi t} + 14400e^{-4\pi t}).\end{aligned}$$

Now, applying the same bounds as in (123) and (124), we can see that:

$$|B(t) - B_0^{(6)}(t)| \leq R_0^{(6)}(t) \text{ for } t \in (0,1]$$

and

$$|B(t) - B_\infty^{(6)}(t)| \leq R_\infty^{(6)}(t) \text{ for } t \in [1,\infty).$$

As before, we can use interval arithmetic to verify that

$$\begin{aligned}&\left|R_0^{(6)}\right| \leq \left|B_0^{(6)}\right| \text{ for } \quad t \in (0,1] \\ &\left|R_\infty^{(6)}\right| \leq \left|B_\infty^{(6)}\right| \text{ for } t \in [1,\infty) \\ &A_0^{(6)}(t) > 0 \text{ for } t \in (0,1] \\ &A_\infty^{(6)}(t) > 0 \text{ for } t \in [1,\infty).\end{aligned}$$

which completes the proof.

□

5. The Case of 24 Dimensions

We have now given a very thorough motivation and treatment of Viazovska's proof that the E_8 lattice gives the densest packing of spheres in 8 dimensions. The argument for 24 dimensions involves almost no new ideas; the same fundamental constructions are used, and they work for exactly the same reasons. Only one section of the proof, the inequality corresponding to Proposition 4.5.2 and 4.5.3, is somewhat more difficult; however, this part of the proof is rather laborious and not particularly enlightening; moreover, much of it leans on computer assistance.

It would be laborious to rehash all of the steps we went through in dimension 8. Thus we will merely give an overview of the argument, and leave the voracious reader to read the paper [8].

5.1. **The +1 Eigenfunction.** We will assume that our +1 function has precisely the same form as it did in 8 dimensions: $\sin^2(\pi r^2/2)$ times the Laplace transform some sort of quasi-modular form. The one difference is to note that, as discussed in section 2.10, the Fourier transform of the 12-dimensional Gaussian has a factor of t^{-12}, so that when we make the $t \mapsto 1/t$ change of variables, a factor of t^{10} will appear where before there was a t^2. (This is the value of $2-n/2$.) Thus we should expect our integrand to be of the form $\phi(-1/z)z^{10}e^{-\pi z r^2}dz$.

The other difference to note is that we will need to eliminate the quadruple zero of the $\sin^2$ factor at $r = 0$, the double zero at $r = \sqrt{2}$ and a single zero at $r = 4$. So our Laplace transform must have a pole of order 4 at $r = 0$, of order 2 at $r = \sqrt{2}$, and of order 1 at $r = 2$. Recall that in dimension 8, our $\sin^2$-times-Laplace-transform expression for the magic function only converged for $r > \sqrt{2}$ due to the presence of the pole at $r = \sqrt{2}$; to define it in general we gave an analytic continuation via a sum of four integrals along shifted contours. In the 24-dimensional case, we should expect our $\sin^2$-times-Laplace-transform expression for the magic function to converge for all $r > 2$. Moreover, we expect the same contour shifting argument to work, as it was designed to change the contour to one invariant under $z \mapsto -1/z$; we still need this behavior in 24 dimensions.

Keeping this in mind, we shall want, for $r > 2$: [48]

$$a(r) := -4\sin^2(\pi r^2/2)\int_0^{i\infty} \phi\left(-\frac{1}{z}\right) z^{10}e^{\pi i z r^2}dz. \tag{126}$$

and, by deforming the contours to those of Figure 6, we shall want to have:

$$\begin{aligned} a(r) := &\int_{-1}^{i} \phi\left(-\frac{1}{z+1}\right)(z+1)^{10}e^{\pi i z r^2}dz \\ &+\int_{1}^{i} \phi\left(-\frac{1}{z-1}\right)(z-1)^{10}e^{\pi i z r^2}dz \\ &-2\int_{0}^{i} \phi\left(-\frac{1}{z}\right)z^{10}e^{\pi i z r^2}dz \\ &+2\int_{i}^{i\infty} \phi(z)\,e^{\pi i z r^2}dz \end{aligned} \tag{127}$$

for all r, where ϕ is a quasimodular form, in analogy with (100). Indeed, given that ϕ is well-behaved, its periodicity is sufficient to make (127) a +1 eigenfunction: applying the

[48]Note that we are replacing the factor of 4 so that the 4's will not bother us in the integral expressions

Fourier transform will exchange the first and last two integrals. In order to make (126) and (127) agree, ϕ will need to satisfy an identity analogous to (67), namely:

$$\Delta^{(2)}\left[\phi\left(-\frac{1}{z}\right)z^{10}\right] = \phi(z). \tag{128}$$

where $\Delta^{(2)}$ is the second finite difference operator. Thus we see that to force this to occur we need $\phi(-1/z)z^{10}$ to expand like in (72):

$$\phi\left(-\frac{1}{z}\right)z^{10} = z^2\phi(z) + z\psi_1(z) + \psi_2(z) \tag{129}$$

which suggests that the weight of our quasimodular form ought to be -8.

Therefore to determine ϕ, we assume that ϕ is some weight 16 polynomial in E_2 E_4 and E_6 divided by Δ^2. We can use (129), as well as consideration of the necessary behavior of the Laplace transform at $r = 0$, $r = \sqrt{2}$ and $r = 2$, to help determine the 5 unknown coefficients of the numerator: those of E_4^4, $E_6^2E_4$, $E_6E_4^2E_2$, $E_4^3E_2^2$ and $E_6^2E_2^2$. We can compute what the numerator ought to be using methods identical to those of the previous chapter. Sparing the details will find that:

$$\begin{aligned}\phi(z) &= \frac{(25E_4^2 - 49E_6^2E_4) + 48E_6E_4^2E_2 + (-49E_4^3 + 25E_6^2)\,E_2^2}{\Delta^2} \\ &= -3657830400q - 314573414400q^2 - 13716864000000q^3 - \cdots\end{aligned} \tag{130}$$

Given this ϕ, we can use exactly the same arguments as in section 4.4. to show that $a(r)$ belongs to Schwartz space and is well-behaved enough for everything to behave as constructed. Indeed $a(r)$ it has a pole of order 4 at $r = 0$, an order 2 pole at $r = \sqrt{2}$, and an order 1 pole at $r = 4$; these account for the only singularities. Moreover, $\phi(-1/z)z^{10}e^{\pi r^2 z}$ and $\phi(z)e^{-\pi r^2 z}$ decay sufficiently quickly as $\mathrm{Im}(z) \to \infty$ for us to be able to deform contours, establishing that (126) and (127) agree. Finally, $a(r)$ it behaves well enough so that the Fourier transform and Laplace transform integrals may be swapped, demonstrating that $a(r)$ is indeed a $+1$ eigenfunction. We can also verify that *ex post* that, when properly normalized so that $a(0) = 1$, $a(r)$ satisfies the Cohn-Miller conjectures.

5.2. **The -1 Eigenfunction.** In analogy with the -1 eigenfunction of Viazovska, we shall let the -1 eigenfunction be the Laplace transform of an actual modular form g_- (as opposed to a quasimodular form as in the $+1$ case). Repeating the same lines of reasoning as we saw in section 4.3, and assuming that $g_-(z)$ has a denominator of Δ^2, we shall see that the numerator of g_- will need to be a holomorphic modular form on $\Gamma(2)$ of weight 14. There is a 7 dimensional space of such forms, but the analogous constraint to (87)[49] narrows the available space down to three dimensions. Examining behavior of $b(r)$ at the poles allows

[49]In this case it is $g(-1/z)z^{10} + g(z-1) = g(z)$.

us to specify the function up to a constant factor, and applying the Cohn-Miller conjectures lets us determine this factor.[50]

We set:

$$b(r) := \int_0^\infty g_-(it)e^{-\pi r^2 t}dz \quad (r > 2) \tag{131}$$

where

$$\begin{aligned} g_-(z) :=& \frac{7\theta_{01}^{20}\theta_{10}^{8} + 7\theta_{01}^{24}\theta_{10}^{24} + 2\theta_{01}^{28}}{\Delta^2} \\ =& 2q^{-2} - 464q^{-1} + 172128 - 367016q^{1/2} \\ &+ 4723846q - 459276288q^{3/2} + O\left(q^2\right). \end{aligned} \tag{132}$$

We analytically continue $b(r)$ to the entire real lines via:

$$\begin{aligned} a(r) : =& \int_{-1}^{i} g_-\left(z+1\right)dz \\ &+ \int_{1}^{i} g_-\left(z-1\right)dz \\ &- 2\int_{0}^{i} g_-\left(z\right)dz \\ &+ 2\int_{i}^{i\infty} g_-\left(-\frac{1}{z}\right)z^{10}dz. \end{aligned} \tag{133}$$

As before, $b(r)$ satisfies all the desired properties. It is a Schwartz function and a -1 eigenfunction of the Fourier transform; both (131) and (131) agree for $r > 4$, and it has a pole of order 2 at $r = 0$, of order 1 at $r = \sqrt{2}$, and of order 1 at $r = 2$.

5.3. **The Final Step.** We now define the 24 dimensional magic function:

$$f(r) = -\frac{\pi i}{113218560}a(r) - \frac{i}{262080\pi}b(r) \tag{134}$$

As in the case of 8 dimensions, we need to show that no extra sign changes occur when we take this sum. Modulo one complication, this once again arises from the fact that the integrand of $f(r) = a(r) + b(r)$ is always less than or equal to 0, and the integrand of $\widehat{f}(r) = a(r) - b(r)$ is always greater than or equal to zero. Indeed, we have:

[50]Unfortunately, as $b(r)$ vanishes at $r = 0$, we cannot merely evaluate the function at 0 to determine this scalar constant.

$$A(t) := \frac{\pi}{28304640} t^{10} \phi(i/t) t^{10} - \frac{1}{65520} g_-(it) \leq 0$$

and

$$B(t) := \frac{\pi}{28304640} t^{10} \phi(i/t) t^{10} + \frac{1}{65520} g_-(it) \geq 0$$

The proofs of these inequalities are similar in idea to those of the dimension 8 case: we expand A and B at "0" and "at ∞", and bound the remainder term to show that these never unexpectedly vanish.

The complication is that the fact that $B(t) \geq 0$ only suffices to show that $\widehat{f}(r) > 0$ for $r > \sqrt{2}$. It does *not* imply that $f(r) \geq 0$ for $0 < r < \sqrt{2}$. Luckily, it is indeed true that $\widehat{f}(r) \geq 0$ for $0 < r < \sqrt{2}$, as can be proved by a more careful analysis of $B(t)$. Thus $f(r)$ satisfies the properties of the 24-dimensional magic function, which proves that the Leech lattice is the densest packing of spheres in 24 dimensions.

For the details of the proofs of these inequalities, we refer the reader to [8].

6. Further Questions and Concluding Remarks

As with many pieces of great mathematics, the solutions to the 8 and 24 dimensional sphere packing problems have inspired several new lines of inquiry. We shall discuss some.

These arguments are not expected to solve the sphere packing problem in any other dimension that 2. Indeed, numerical evidence shows that the Cohn-Elkies bound is somewhat larger than the density of known sphere packings in dimensions other than 1, 2, 8 ans 24; moreover, following the analogy with Delsarte's bounds for kissing numbers, it seems quite likely that dimensions 8 and 24 will be amenable to techniques that fail in other dimensions. Indeed, 8 and 24 dimensions seem to be the great exceptions when it comes to all things sphere packing.

However, there is an interesting question of constructing the magic function in dimension 2. [51] This would supply a new proof of Thue's theorem, and would require a somewhat different construction than Viazovka's. A magic function for dimension 2 must vanish to order one at 1, and to order two at all higher magnitudes of the A_2 lattice; that is, numbers of the form $\sqrt{2\left(a^2+ab+b^2\right)}$. A classical theorem in number theory says that numbers k of the form a^2+ab+b^2 are precisely those integers for which all primes congruent to -1 modulo 3 appear to an even power in k's prime factorization. The number of such magnitudes less than N grows like $C \cdot N/\sqrt{\log(N)}$ for a constant C. This suggests that the magic function f is not as constrained as it is in dimensions 8 and 24; this, along with observed phenomenon that different numerical approximations for optimal functions in dimension 2 appear to sometimes converge to distinct functions, leads one to conclude that the magic function in dimension 2 is not unique.

[51] One could also ask the (presumably easier) question of whether there is a 1-dimensional magic function that resembles Viazovska's solution in dimension 8. While several 1-dimensional magic functions are now known, none appear to involve modular forms as we saw with Viazovska's work.

As for a construction of a 2 dimensional magic function, several difficulties arise if we wish to proceed along the lines of Viazovska. Introducing a factor like $\sin^2$ to force the roots into place would likely introduce too many zeroes; on the other hand, a function that vanishes only at the magnitudes listed above could not be periodic, and so could not have a q series. This deprives us of one of the main features of Viazovska's argument. Dimension 2 poses another interesting challenge: A_2 is not self-dual; A_2^* is a scaled copy of A_2. This situation is quite different from E_8 and the Leech lattice, both of which are unimodular.

Some further questions concern the nature of the (now known) 8 and 24 dimensional magic functions themselves. For instance, we can ask: for which $r \in \mathbb{C}$ does the magic function converge? Where does it accumulate essential singularities? Similarly, we could ask about where the magic functions vanish (outside of the real line, of course, where we have what we might call "trivial zeroes"). As discussed in section 3.3, numerical experimentation seemed to reveal that the magic function approximations in [9] accumulated a dense line of non-real roots in $\mathbb{C}$, forming what appeared to be a natural boundary. Additionally, there appeared to be a non-real root on the imaginary axis for both the 8 and 24 dimensional magic function (though different actual values for each).

In fact, the convergence of the approximations used to guess the magic functions is itself of interest. As we have discussed, various ad-hoc methods seem to converge to the magic functions in dimensions 8 and 24, while some plausible methods of approximation methods ultimately fail (even after initially appearing to work). It would be interesting to get to the bottom of these phenomena.

Despite the fact that the Cohn-Elkies linear programming bound seems unable to provide us with optimal bounds for sphere packings in dimensions other than 1, 2, 8 and 24, it is still interesting to examine what the optimal functions for the linear programming bound might be in other dimensions. This question is of interest for its own sake (it helps give us some sense about how much we can simultaneously control the sign of f and $\widehat{f}$ in various dimensions), and because it provides us with bounds on sphere packing densities for all dimensions d. The question of the asymptotics of the Cohn-Elkies bound as $d \to \infty$ is still quite mysterious.

It is also rather mysterious that the Cohn-Elkies bound *does* give optimal bounds in dimensions 8 and 24, but, apparently for no other dimension above 2. The lattices giving the densest packings in 8 and 24 dimensions are themselves highly unusual objects; indeed, they are really the archetypal "exceptional structures" of mathematics: E_8 is the root lattice of the largest exceptional semisimple Lie algebra, and the Leech lattice is closely related to several exceptional simple finite groups, including the famous "monster group" (which was originally discovered in connection with the Leech lattice). Yet the magic functions seem to not utilize any of the extraordinary structure that these lattices possess. Perhaps there are deeper connections between the magic functions and the E_8 and Leech lattices; perhaps such connections will help shed light on *why* the Cohn-Elkies bound is optimal in dimensions 8 and 24 but for no other dimensions greater than 2.[52]

In a somewhat different vein, the discovery of the 8 and 24 dimensional magic functions and their apparent uniqueness have led to new work on "Fourier interpolation" – the question of the extent to which one can simultaneously fix the values of f and $\widehat{f}$, and how this constrains the function f. Viazovska, Cohn, Kumar, Miller and Radchenko have conjectured that, given specified values of $f\left(\sqrt{2n}\right)$, $f'\left(\sqrt{2n}\right)$, $\widehat{f}\left(\sqrt{2n}\right)$, and $\widehat{f'}\left(\sqrt{2n}\right)$ for positive integers n (subject to certain clearly necessary constraints, like a sufficiently rapid rate of decay as $n \to \infty$), then there exists a unique radial Schwartz function f interpolating these values. Moreover, forcing many of these values to be 0 does give rise to functions that appear related to modular forms.[53]

While the conjecture concerning the interpolation of the values of f, f', $\widehat{f}$, and $\widehat{f'}$ at $\sqrt{2n}$ for a radial Schwartz function f is still work in progress, Viazovska and Radchenko have given a complete characterization of the values that an even 1-dimensional Schwartz function f and its Fourier transform $\widehat{f}$ can take at the values $\pm\sqrt{n}$. Rather wonderfully, it turns out that the only constraint is the "obvious one" that comes from Poisson summation. Namely, if x_n and y_n are two sequences of rapid decay (that is, such that for all $k > 0$, $n^k x_n \to 0$ and $n^k y_n \to 0$ as $n \to \infty$) and these two sequences satisfy $\sum_{n=0}^{\infty} x_{n^2} = \sum_{n=0}^{\infty} y_{n^2}$, then there exists a unique even Scwhartz function f such that $f(\pm\sqrt{n}) = x_n$, $\widehat{f}(\pm\sqrt{n}) = y_n$. Moreover, this function can be explicitly constructed using modular forms much like the magic functions in dimension 8 and 24 [32].

There is also some speculation as to whether a modified version of the Cohn-Elkies linear programming bound may lead to solutions of the sphere packing problem in other dimensions. In particular, like with Musin's resolution of the 4-dimensional kissing number problem, there is hope for a resolution of the 4-dimensional sphere packing problem, where the optimal packing is almost certainly the lattice D_4. A potential candidate for stronger bounds are the "semi-definite programming bounds" of de Laat and Vallentin [20]. These lie in a hierarchy, the first level of which – the so-called "two-point bound" – is exactly the Cohn-Elkies linear programming bound. It is conceivable that higher levels of this hierarchy will yield optimal bounds in dimension 4. (In dimension 3, however, many periodic packings attain maximal deinsity, and a test function would have to satisfy the optimality conditions A) – C) of section 3.1 for each such periodic packing, so it seems that only a radical departure from the Cohn-Elkies bound could be used to give a new solution to the Kepler conjecture.)

We conclude with some general remarks. The resolution of the 8 and 24 dimensional sphere packing problems gives us a complete, elegant solution to two cases of one of the most natural but difficult questions in geometry. And these results have at last demonstrated rigorously one of the most fundamental properties of the exceptional E_8 and Λ_{24} lattices. The arguments involve no laborious case-by-case verification, and use techniques that are

[52]However, while the numerical evidence is extremely convincing, it is worth noting that it is not actually known that the Cohn-Elkies bound fails to yield optimal sphere packing bounds in other dimensions. As Cohn points out in [4], so great is our ignorance on this issue that we cannot even rule out the possibility that the Cohn-Elkies bound to gives optimal bounds for all sufficiently high dimensions (though, as Cohn says, "that's clearly ridiculous").

[53]This I have learned from private correspondence with Henry Cohn.

essentially classical in nature.[54] Finally, these proofs offer yet another testament to the versatility and power of modular forms. Many have pointed out the seemingly "unreasonable" effectiveness of modular forms in mathematics; with Viazovska's work, it should now be clear that they are, in fact, magic.

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