Sample Questions from Past Qualifying Exams

This list may give the impression that the exams consist of a series of questions fired at the student one after another. In fact most exams have more the character of a conversation with considerable give and take. Hence this list cannot be expected to indicate accurately the difficulties involved.

Foundations

- Wxplain to a mathemtician what you mean by a sentence (for example, in the language of groups). Explain the syntax and semantics of this language.
- What does the completeness theorem tell us about the sentences in this language? What about arbitrary sets of sentences?
- What does the completeness theorem say aobut the valid sentences of number theory?
- What can you say about the axiom system P of number theory?
- What does the undecidability of Q tell you about the valid sentences in the language of number theory?
- What is meant by a relation definable in the standard model of number theory?
- Are all arithmetical relations Σ_1 definable? Why?
- What does the completeness tell a group theorist?
- Let σ be a first order sentence. Suppose a group theorist has proved (in some fashion, not necessarily first order) that σ is true of all torsion free groups. What can you tell him?
- Does the completeness theorem tell us taht there is a way for us to decide, given a first order sentence, whether it is true for all group?
- Is ZF decidable? Is the consistency of P decidable in ZF?
- What is Church's thesis? Why can't it be proved?
- What is Beth's Theorem? Sketch a proof.
- Is the complete theory of $\langle \omega, +, \cdot \rangle$ model copmlete?
- What is the Compactness Theorem? How is it proved?
- What sets are definable in $\langle \omega, \mathrm{S} \rangle$? What do you know about $\langle \omega, \mathrm{S} \rangle$? Is it finitely axiomatizable?
- What sets are definable in $\langle \omega, \mathrm{S} \rangle$ in second order logic?
- In what sense is every first order sentence equivalent to a Σ sentence? A $\forall\exists$ sentence?
  - Does the above require the Axiom of Choice?
- Is the second order theory of $\langle \omega, \mathrm{S} \rangle$ decidable?
- Do you know a *natural* essentially undecidable finitely axiomatizable subtheory of ZF?
- Do you know a sytactical equivalent to model cmpleteness?
- What decidable theories do you know?
- How do you prove Gödel's Incompleteness Theore?

- Is ZF finitely axiomatizable?
- Is there a theory T such that $\mathcal{A}$ is a model of T iff $\mathcal{A}$ is a finite group?
- Is there a theory T such taht a sentence σ holds in T if and only if σ holds in every finite group?
- What important properties does Q have? (The theory Q of Tarski's *Undec. Theories*).
- Prove that if a theory T has no complete axiomatizable extension, tehn T is undecidable.
- Do you know any theories with denumerably ennumerable models?
- Give your favorite system of propositional calculus, and show that this system is uniquely readble. Give a procedure for reading formulas.
- Let L be the language with $0,\ s,\ +,\ \cdot$. Let T be the theory in L whose only non-logical axiom is $s_x \cdot s_y = x \cdot y + (x + s_y)$. Is T decidable?
- Is there a theory in which the Gödel sentence expressing inconsistency is true, and yet which is consistent? Why?
- Use the Completeness Theorem to prove the existence of non-Archimedean fields.
- State the downward Löwenheim-Skolem Theorem.
- Give an example to show why it is important to have "elementary substructure" and not just "elementary equivalent structure" in the downward Löwenheim-Skolem theorem.
- Define recursive function, recursive set, and recursevely ennumerable set.
- What is the most important result about representability and what is it used for?
- Is there an undecidable theory with just one mathematical symbol?
- Do you know anything interesting about propositional calculus?
- What is the most general form you know of Gödel's Theorem?
- Name a decidable theory. How do you know it is decidable?
- What is meant by a proof?
- What is meant by a categorical theory? Give an example.
- Can one define $\vee$ in terms of $\rightarrow$?
- What is the relation between completeness and compactness?
- Is the set of Σ_1 sentences true in $\langle \omega, +, \cdot \rangle$ recursively ennumerable? Is it recursive?
- What is the difference between recursive and primitive recursive?
- Give a purely algebraic characterization of EC_Δ classes.
- What is an elementary class? [**Silver**]
  - Are the cyclic groups an elementary class?
  - Are the torsion groups an elementary class?

- How do you prove that the theory of algebraically closed fields of characteristic zero is decidable? [**Silver**]
- Why is a complete axiomatizable theory decidable? [**Silver**]
- What is Gödel's Second Incompleteness Theorem? [**Silver**]

Set Theory

- Are there measurable ordinals?
- Draw a Venn diagram showing recursive, recursively ennumerable, co-recursively ennumerable, primite recursive, arithemtical, second order definable sets, and implicitly definable sets. State Beth's Theorem, and show where the following sets fall, and explain why:
  - $\mathrm{Th}(\omega, +)$, $\mathrm{Th}(\omega, \mathrm{S})$.
  - Sets definable in second order theory of $\langle \omega, + \rangle$.
  - $(\mathbb{Q}, +, \cdot)$, $(\omega, +, \cdot)$.
  - Universal validities and existential validities
  - Theory of groups; universal sentences of group theory.
  - $\forall\exists$ sentences.
  - $\mathrm{Th}(\mathbb{R}, +, \cdot)$, $\mathrm{Th}(\mathbb{Z}, +\cdot)$.
- Prove the recursion theorem and Rice's theorem.
- Show that any uncountable well-ordered set has a countable well-ordered subset.
- Define "representable set".