QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Tuesday, March 12 (Day 1)

1. Let X be a compact n-dimensional differentiable manifold, and $Y \subset X$ a closed submanifold of dimension m. Show that the Euler characteristic $\chi(X \setminus Y)$ of the complement of Y in X is given by

$$\chi(X \setminus Y) \;=\; \chi(X) + (-1)^{n-m-1}\chi(Y).$$

Does the same result hold if we do not assume that X is compact, but only that the Euler characteristics of X and Y are finite?

2. Prove that the infinite sum

$$\sum_{p \text{ prime}} \frac{1}{p} \;=\; \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots$$

diverges.

3. Let $h(x)$ be a $\mathcal{C}^\infty$ function on the real line $\mathbb{R}$. Find a $\mathcal{C}^\infty$ function $u(x,y)$ on an open subset of $\mathbb{R}^2$ containing the x-axis such that

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} \;=\; u^2$$

and $u(x,0) = h(x)$.

4. a) Let K be a field, and let $L = K(\alpha)$ be a finite Galois extension of K. Assume that the Galois group of L over K is cyclic, generated by an automorphism sending α to $\alpha + 1$. Prove that K has characteristic $p > 0$ and that $\alpha^p - \alpha \in K$.

b) Conversely, prove that if K is of characteristic p, then every Galois extension L/K of degree p arises in this way. (Hint: show that there exists $\beta \in L$ with trace 1, and construct α out of the various conjugates of β.)

5. For small positive α, compute

$$\int_0^\infty \frac{x^\alpha \, dx}{x^2 + x + 1}.$$

For what values of $\alpha \in \mathbb{R}$ does the integral actually converge?

6. Let $M \in \mathcal{M}_n(\mathbb{C})$ be a complex $n \times n$ matrix such that M is similar to its complex conjugate $\overline{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that M is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.

QUALIFYING EXAMINATION
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Wednesday, March 13 (Day 2)

1. Prove the Brouwer fixed point theorem: that any continuous map from the closed n-disc $D^n \subset \mathbb{R}^n$ to itself has a fixed point.

2. Find a harmonic function f on the right half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ satisfying

$$\lim_{x \to 0+} f(x+iy) = \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \end{cases} .$$

3. Let n be any integer. Show that any odd prime p dividing n^2+1 is congruent to 1 (mod 4).

4. Let V be a vector space of dimension n over a finite field with q elements.
   a) Find the number of one-dimensional subspaces of V.
   b) For any $k : 1 \leq k \leq n-1$, find the number of k-dimensional subspaces of V.

5. Let K be a field of characteristic 0. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials $F(X,Y,Z)$ of degree d modulo scalars ($N = d(d+3)/2$). Let $W \subset \mathbb{P}^N$ be the subset of polynomials F of the form

$$F(X,Y,Z) \ = \ \prod_{i=1}^{d} L_i(X,Y,Z)$$

for some collection of linear forms $L_1, \ldots, L_d$.
   a. Show that W is a closed subvariety of $\mathbb{P}^N$.
   b. What is the dimension of W?
   c. Find the degree of W in case $d = 2$ and in case $d = 3$.

6. a. Suppose that $M \to \mathbb{R}^{n+1}$ is an embedding of an n-dimensional Riemannian manifold (i.e., M is a hypersurface). Define the *second fundamental form* of M.

b. Show that if $M \subset \mathbb{R}^{n+1}$ is a compact hypersurface, its second fundamental form is positive definite (or negative definite, depending on your choice of normal vector) at at least one point of M.

QUALIFYING EXAMINATION
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Thursday, March 14 (Day 3)

1. In $\mathbb{R}^3$, let S, L and M be the circle and lines

$$\begin{aligned} S &= \{(x,y,z) : x^2+y^2=1;\ z=0\} \\ L &= \{(x,y,z) : x=y=0\} \\ M &= \{(x,y,z) : x=\frac{1}{2};\ y=0\} \end{aligned}$$

respectively.

a. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L)$.

b. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L \cup M)$.

2. Let $L, M, N \subset \mathbb{P}^3_{\mathbb{C}}$ be any three pairwise disjoint lines in complex projective threespace. Show that there is a unique quadric surface $Q \subset \mathbb{P}^3_{\mathbb{C}}$ containing all three.

3. Let G be a compact Lie group, and let $\rho : G \to GL(V)$ be a representation of G on a finite-dimensional $\mathbb{R}$-vector space V.

a) Define the *dual representation* $\rho^* : G \to GL(V^*)$ of V.

b) Show that the two representations V and V^* of G are isomorphic.

c) Consider the action of $SO(n)$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and the corresponding representation of $SO(n)$ on the vector space V of $\mathcal{C}^\infty$ $\mathbb{R}$-valued functions on S^{n-1}. Show that each nonzero irreducible $SO(n)$-subrepresentation $W \subset V$ of V has a nonzero vector fixed by $SO(n-1)$, where we view $SO(n-1)$ as the subgroup of $SO(n)$ fixing the vector $(0, \ldots, 0, 1)$.

4. Show that if K is a finite extension field of $\mathbb{Q}$, and A is the integral closure of $\mathbb{Z}$ in K, then A is a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$ (the degree of the field extension). (Hint: sandwich A between two free $\mathbb{Z}$-modules of the same rank.)

5. Let n be a nonnegative integer. Show that

$$\sum_{\substack{0\le k\le l\\ k+l=n}} (-1)^l \binom{l}{k} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ -1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } n \equiv 2 \pmod 3 \end{cases}.$$

(Hint: Use a generating function.)

6. Suppose K is integrable on $\mathbb{R}^n$ and for $\epsilon > 0$ define

$$K_\epsilon(x) = \epsilon^{-n} K(\frac{x}{\epsilon}).$$

Suppose that $\int_{\mathbb{R}^n} K = 1$.

a. Show that $\int_{\mathbb{R}^n} K_\epsilon = 1$ and that $\int_{|x|>\delta} |K_\epsilon| \to 0$ as $\epsilon \to 0$.

b. Suppose $f \in L^p(\mathbb{R}^n)$ and for $\epsilon > 0$ let $f_\epsilon \in L^p(\mathbb{R}^n)$ be the convolution

$$f_\epsilon(x) \; = \; \int_{y\in\mathbb{R}^n} f(y) K_\epsilon(x-y) dy\,.$$

Show that for $1 \le p < \infty$ we have

$$\|f_\epsilon - f\|_p \to 0 \text{ as } \epsilon \to 0.$$

c. Conclude that for $1 \le p < \infty$ the space of smooth compactly supported functions on $\mathbb{R}^n$ is dense in $L^p(\mathbb{R}^n)$.

Extra problems: Let me know if you think these should replace any of the ones above, either for balance or just by preference.

1. Suppose that $M \to \mathbb{R}^N$ is an embedding of an n-dimensional manifold into N-dimensional Euclidean space. Endow M with the induced Riemannian metric. Let $\gamma : (-1,1) \to M$ be a curve in M and $\overline{\gamma} : (-1,1) \to \mathbb{R}^N$ be given by composition with the embedding. Assume that $\|\frac{d\overline{\gamma}}{dt}\| \equiv 1$. Prove that γ is a geodesic iff

$$\frac{d^2\overline{\gamma}}{dt^2}$$

is normal to M at $\gamma(t)$ for all t.

2. Let A be a commutative Noetherian ring. Prove the following statements and explain their geometric meaning (even if you do not prove all the statements below, you may use any statement in proving a subsequent one):

a) A has only finitely many minimal prime ideals $\{\mathbf{p_k} \mid k = 1, \ldots, n\}$, and every prime ideal of A contains one of the $\mathbf{p_k}$.

b) $\bigcap_{k=1}^n \mathbf{p_k}$ is the set of nilpotent elements of A.

c) If A is reduced (i.e., its only nilpotent element is 0), then $\bigcup_{k=1}^n \mathbf{p_k}$ is the set of zero-divisors of A.

4. Let A be the $n \times n$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1/n & 1/n & 1/n & \ldots & 1/n \end{pmatrix}.$$

Prove that as $k \to \infty$, A^k tends to a projection operator P onto a one-dimensional subspace. Find the kernel and image of P.