QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 20, 2015 (Day 1)

1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d.

(a) Let K_C be the canonical bundle of C. For what integer n is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then C is not hyperelliptic.

(c) Prove that if $d \geq 5$ then C is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

Solution: By the adjunction formula, the canonical divisor class is $K_C = \mathcal{O}_C(d-3)$, that is, plane curves of degree $d-3$ cut out canonical divisors on C. It follows that if $d \geq 4$ then any two points $p, q \in C$ impose independent conditions on the canonical series $|K_C|$; that is, $h^0(K_C(-p-q)) = g-2$, so by Riemann-Roch $h^0(\mathcal{O}_C(p+q)) = 1$, i.e., C is not hyperelliptic. Similarly, if $d \geq 5$ then any three points $p, q, r \in C$ impose independent conditions on the canonical series $|K_C|$; by Riemann-Roch it follows that $h^0(\mathcal{O}_C(p+q+r)) = 1$ so C is not trigonal.

2. (A) Let S_4 be the group of automorphisms of a 4-element set. Give the character table for S_4 and explain how you arrived at it.

Solution: To start with, there are five conjugacy classes in S_4: $(1), (12), (123), (1234)$ and $(12)(34)$. The characters of the trivial and alternating representations U and U' are clear. The standard representation of S_4 on $\mathbb{C}^4$ splits as a direct sum of the trivial and a three-dimensional representation V, whose character is simply the character of $\mathbb{C}^4$ minus one; we see that it's irreducible because the norm of its character is 1. We get another irreducible as $V' = V \otimes U'$; its character is $\chi_{V'} = \chi_V \chi_{U'}$. The final irreducible representation W (and its character) can be found by pulling back the standard representation of S_3 via the quotient map $S_4 \to S_3$ (or by the orthogonality relations). Altogether, we have

conjugacy class	e	(12)	(123)	(1234)	(12)(34)
number of elements	1	6	8	6	3
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	1	0	2

3. (DG) Let

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - z^3 - z = 0\}.$$

(a) Prove that M is a smooth surface in $\mathbb{R}^3$.

(b) For what values of $c \in \mathbb{R}$ does the plane $z = c$ intersect M transversely?

Solution: See attached.

4. Define the Banach space $\mathcal{L}$ to be the completion of the space of continuous functions on the interval $[-1, 1] \subset \mathbb{R}$ using the norm

$$||f|| = \int_{-1}^{1} |f(t)|dt.$$

Suppose that $f \in \mathcal{L}$ and $t \in [-1, 1]$. For $h > 0$, let I_h be the set of points in $[-1, 1]$ with distance h or less from t. Prove that

$$\lim_{h \to 0} \int_{t \in I_h} |f(t)|dt = 0$$

Solution: See attached.

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,

(a) with coefficients in $\mathbb{Z}$?

(b) with coefficients in $\mathbb{Z}/2$?

(c) with coefficients in $\mathbb{Z}/3$?

Solution: $\mathbb{RP}^2$ and $\mathbb{RP}^3$ have cell complexes with sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

where the maps are alternately 0 and multiplication by 2; from this the homology groups of $\mathbb{RP}^2$ and $\mathbb{RP}^3$ can be calculated as $\mathbb{Z}, \mathbb{Z}/2, 0$ and $\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$ respectively. The rest is just Kunneth; the answers are

(a): $\mathbb{Z}$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}$, $\mathbb{Z}/2$, 0;

(b): $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}/2$,

(c): $\mathbb{Z}/3$, 0, 0, $\mathbb{Z}/3$, 0, 0

6. Let Ω be an open subset of the Euclidean plane $\mathbb{R}^2$ A map $f : \Omega \to \mathbb{R}^2$ is said to be *conformal* at $p \in \Omega$ if its differential df_p preserves the angle between any two tangent vectors at p. Now view $\mathbb{R}^2$ as $\mathbb{C}$ and a map $f : \Omega \to \mathbb{R}^2$ as a $\mathbb{C}$-valued function on Ω.

(a) Supposing that f is a holomorphic function on Ω, prove that f is conformal where its differential is nonzero.

(b) Suppose that f is a nonconstant holomorphic function on Ω, and $p \in \Omega$ is a point where $df_p = 0$. Let L_1 and L_2 denote distinct lines through p. Prove that the angle at $f(p)$ between $f(L_1)$ and $f(L_2)$ is n times that between L_1 and L_2, with n being an integer greater than 1.

Solution: See attached.

QUALIFYING EXAMINATION

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Wednesday January 21, 2015 (Day 2)

1. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2+y^2+z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$.

(a) What are the homology groups of X?

(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

Solution: Under the attaching map $I \hookrightarrow X$, the boundary $\varphi(I)$ is homologous to 0, so attaching I simply adds one new, non-torsion generator to H^1; thus

$$H_0(X) = H^1(X) = H^2(X) = \mathbb{Z},$$

and all other homology groups are 0. Similarly, $\pi_1(X) = \mathbb{Z}$. For $\pi_2(X)$, note that the universal cover of X is a string of spheres attached in a sequence by line segments; $\pi_2(X)$ is thus the free abelian group on countably many generators.

2. (A) Let

$$f(t) = t^4 + bt^2 + c \in \mathbb{Z}[t].$$

(a) If E is the splitting field for f over $\mathbb{Q}$, show that $Gal(E/\mathbb{Q})$ is isomorphic to a subgroup of the dihedral group D_8.

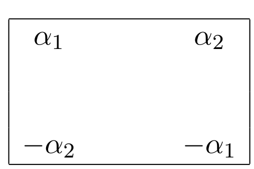
(b) Given an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Justify.

(c) Give an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Justify.

(d) Give an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to D_8.

Solution:

(a) Obviously if α is a root of f, so is $-\alpha$. So let $\pm\alpha_1, \pm\alpha_2$ be the four distinct roots of f in E. If ϕ is an element of the Galois group, it must permute the roots of f—moreover, ϕ is determined completely by its action on α_1 and α_2. Also by definition of automorphism, note that $\phi(\alpha_1)$ cannot be a rational multiple of $\phi(\alpha_2)$, while $\phi(-\alpha_1) = -\phi(\alpha_1)$. Hence any field automorphism must necessarily give rise to a symmetry of the following square:



This gives the injection of Gal into D_8.

(b) An obvious strategy is to find a quadratic extension of a quadratic extension, then find an element whose minimal polynomial is degree 4. For instance, the element $\alpha = \sqrt{2} + \sqrt{3}$ in $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ has a degree 4 minimal polynomial which we can construct by repeatedly multiplying by conjugates: Begin with $t - \sqrt{2} - \sqrt{3}$, the multiply by $(t - \sqrt{2}) + \sqrt{3}$, then multiply this by $(t^2 + \sqrt{2})^2 - 3$. For this choice of α, we have $f(t) = t^4 - 10t^2 + 1$.

(c) Taking $b = 0$ and $c = 1$, we see that the splitting field is isomorphic to the subfield of the complex numbers generated by adjoining to $\mathbb{Q}$ the number $\alpha = e^{\pi i/4}$. This is a degree 4 field over $\mathbb{Q}$. Since we have a splitting field in characteristic zero, the Galois group has order 4. We see that the field automorphism sending

$$\alpha \mapsto \alpha^3$$

has order 4, hence the Galois group is cyclic.

(d) Take $b = 0$ and $c = 2$. Clearly we have roots $\alpha_1 = 2^{1/4}$ and $\alpha_2 = i2^{1/4}$, which together lie in an extension of at least degree 8 over $\mathbb{Q}$. By part (a), the Galois group must be D_8 itself.

3. (CA) Let $a \in (0, 1)$. By using a contour integral, compute

$$\int_0^{2\pi} \frac{dx}{1 - 2a\cos x + a^2}.$$

Solution (HT): By the periodicity of cos, it suffices to compute the integral from $-\pi$ to π. We note that there is a pole for the function

$$f(z) = \frac{1}{1 - 2a\cos z + a^2}$$

at $z_0 = i\cosh^{-1}\frac{1+a^2}{2a}$. Let R_t be the rectangle bordered by the lines $x = \pm\pi$ and $y = 0, y = t$. As $t \to \infty$, the contribution from the line $y = t$ goes to zero. On the other hand, for all values of t, the contribution to the integral from $x = \pm\pi$ cancel each other out. Thus the integral along the bottom edge of the rectangle (which is what we seek) is equal to $2\pi i$ times the residue of $f(z)$ at z_0. Near z_0, we have that

$$1 - 2a\cos z + a^2 = (z - z_0)2ai\sinh iz_0 + \dots$$

so we conclude the integral is given by

$$\frac{2\pi i}{2ai\sinh z_0}.$$

This simplifies to

$$\frac{2\pi}{1 - a^2}.$$

Alternate solution (CH): Write the integral as a contour integral on the unit circle: set $dx = \frac{-idz}{z}$, so that

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos x + a^2} dx = -i \int_{|z|=1} \frac{1}{z(1+a^2) - az^2 - a} dz.$$

Factor the denominator to find the poles of the latter integrand; one is inside the unit circle and one outside. Calculate the residue at the former pole and use Cauchy's theorem to evaluate the integral.

4. (AG) Let K be an algebraically closed field of characteristic 0 and let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface over K.

(a) Show that Q is rational by exhibiting a birational map $\pi : Q \to \mathbb{P}^{n-1}$.

(b) How does the map π factor into blow-ups and blow-downs?

Solution: For the first part, we choose any point $p \in Q$ and take π to be the projection from p. Since Q has degree 2, a general line in $\mathbb{P}^n$ through p will meet Q in one other point, so that the map $\pi : Q \to \mathbb{P}^{n-1}$ has degree 1; that is, it is a birational map. This map blows up the point p, and then blows down the union of the lines on Q through p. In the other direction, starting with $\mathbb{P}^{n-1}$ we blow up the intersection $Z = S \cap H$ of a quadric hypersurface $S \subset \mathbb{P}^{n-1}$ and a hyperplane $H \subset \mathbb{P}^{n-1}$, and then blow down the proper transform of H.

5. DG Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

be the unit sphere centered at the origin in $\mathbb{R}^3$.

(a) Prove that the vector field

$$v = yz\frac{\partial}{\partial x} + zx\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial z}$$

on $\mathbb{R}^3$ is tangent to S at all points of S, and thus defines a section of the tangent bundle TS.

(b) Let g be the metric on S induced from the euclidean metric on $\mathbb{R}^3$, and let ∇ be the associated, metric compatible, torsion free covariant derivative. The tensor ∇v is a section of $TS \otimes TS^*$. Write ∇v at the point $(0, 0, 1) \in S$ using the coordinates (x_1, x_2) given by the map $(x_1, x_2) \mapsto (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$ from the unit disc $x_1^2 + x_2^2 < 1$ to S.

Solution: See attached

6. (RA) Let L be a positive real number.

(a) Compute the Fourier expansion of the function x on the interval $[-L, L] \subset \mathbb{R}$.

(b) Prove that the Fourier transform does not converge to x pointwise on the closed interval $[-L, L]$.

Solution: See attached. One note: the second part follows immediately from the observation that whatever the Fourier expansion converges to at $-L$ must be the same as what it converges to at L.

QUALIFYING EXAMINATION

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Thursday January 22, 2015 (Day 3)

1. (DG) The helicoid is the parametrized surface given by

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \to (v \cos u, v \sin u, au)$$

where a is a real constant. Compute its induced metric.

Solution. Compute $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ and deduce that the metric is $g = (v^2 + a^2) du \otimes du + dv \otimes dv$.

2. (RA) A real valued function defined on an interval $(a, b) \subset \mathbb{R}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

whenever $x, y \in (a, b)$ and $t \in (0, 1)$.

(a) Give an example of a non-constant, non-linear convex function.

(b) Prove that if f is a non-constant convex function on $(a, b) \in \mathbb{R}$, then the set of local minima of f is a connected set where f is constant.

Solution: See attached

3. (AG) Let K be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree n in two variables over K. Let $X \subset \mathbb{P}^n$ be the locus of n^{th} powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that X and $Y \subset \mathbb{P}^n$ are closed subvarieties.

(b) What is the degree of X?

(c) What is the degree of Y?

Solution: First, X is the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ sending $[a, b] \in \mathbb{P}^1$ to $(ax + by)^n \in \mathbb{P}^n$. This is projectively equivalent (in characteristic 0!) to the degree n Veronese map, whose image is a closed curve of degree n. Second, Y is the zero locus of the discriminant, which is a polynomial of degree $2n - 2$ in the coefficients of a polynomial of degree n (this number can be deduced from the Riemann-Hurwitz formula, which says that a degree n map from $\mathbb{P}^1$ to $\mathbb{P}^1$ has $2n - 2$ branch points; that is, a general line in $\mathbb{P}^n$ meets Y in $2n - 2$ points).

4. (AT) Let X be a compact, connected and locally simply connected Hausdorff space, and let $p : \tilde{X} \to X$ be its universal covering space. Prove that $\tilde{X}$ is compact if and only if the fundamental group $\pi_1(X)$ is finite.

Solution: See attached

5. (CA) Prove that if f and g are entire holomorphic functions and $|f| \leq |g|$ everywhere, then $f = \alpha \cdot g$ for some complex number α.

Solution: The conclusion trivially holds in the case $g = 0$; from now on, assume that g is not the zero function. The identity theorem implies that the zeros of g are isolated, so $h := f/g$ is meromorphic. The function h is bounded by hypothesis, so Riemann's theorem implies that h can be extended to an entire bounded function. Liouville's theorem implies that h is constant, which implies the conclusion.

6. (A) Consider the rings

$$R = \mathbb{Z}[x]/(x^2+1) \quad \text{and} \quad S = \mathbb{Z}[x]/(x^2+5).$$

(a) Show that R is a principal ideal domain.

(b) Show that S is not a principal ideal domain, by exhibiting a non-principal ideal.

Solution: For the first, the fact that R is a principal ideal domain follows from the fact that it's a Euclidean domain, with size function $|z|^2$: for any $a, b \in R$ we can write

$$b = ma + r$$

with $|r| < |a|$; carrying this out repeatedly shows that the ideal generated by two elements of R can be generated by one. For the second, the ideal $(2, 1+x) \subset S$ is not principal.