QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 31, 2010 (Day 1)

1. (CA) Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx.$$

2. (A) Let b be any integer with $(7, b) = 1$ and consider the polynomial

$$f_b(x) = x^3 - 21x + 35b.$$

(a) Show that f_b is irreducible over $\mathbb{Q}$.

(b) Let P denote the set of $b \in \mathbb{Z}$ such that $(7, b) = 1$ and the Galois group of f_b is the alternating group A_3. Find P.

3. (T) Let X be the Klein bottle.[1]

(a) Compute the homology groups $H_n(X, \mathbb{Z})$.

(b) Compute the homology groups $H_n(X, \mathbb{Z}/2)$.

(c) Compute the homology groups $H_n(X \times X, \mathbb{Z}/2)$.

4. (RA) Let f be a Lebesgue integrable function on the closed interval $[0, 1] \subset \mathbb{R}$.

(a) Suppose that g is a continuous function on $[0, 1]$ such that the integral of $|f - g|$ is less than ϵ^2. Prove that the set where $|f - g| > \epsilon$ has measure less than ϵ.

(b) Show that for every $\epsilon > 0$, there is a continuous function g on $[0, 1]$ such that the integral of $|f - g|$ is less than ϵ^2.

5. (DG) Let v denote a vector field on a smooth manifold M and let $p \in M$ be a point. An *integral curve* of v through p is a smooth map $\gamma : U \to M$ from a neighborhood U of $0 \in \mathbb{R}$ to M such that $\gamma(0) = p$ and the differential $d\gamma$ carries the tangent vector $\partial/\partial t$ to $v(\gamma(t))$ for all $t \in U$.

(a) Prove that for any $p \in M$ there is an integral curve of v through p.

(b) Prove that any two integral curves of v through any given point p agree on some neighborhood of $0 \in \mathbb{R}$.

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[1]The Klein bottle is obtained from the square $I^2 = \{(x, y) : 0 \le x, y \le 1\} \subset \mathbb{R}^2$ by the equivalence relation $(0, y) \sim (1, y)$ and $(x, 0) \sim (1 - x, 1)$

(c) A *complete* integral curve of v through p is one whose associated map has domain the whole of $\mathbb{R}$. Give an example of a nowhere zero vector field on $\mathbb{R}^2$ that has a complete integral curve through any given point. Then, give an example of a nowhere zero vector field on $\mathbb{R}^2$ and a point which has no complete integral curve through it.

6. (AG) Show that a general hypersurface $X \subset \mathbb{P}^n$ of degree $d > 2n - 3$ contains no lines $L \subset \mathbb{P}^n$.

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Wednesday September 1, 2010 (Day 2)

1. (T) If M_g denotes the closed orientable surface of genus g, show that continuous maps $M_g \to M_h$ of degree 1 exist if and only if $g \geq h$.

2. (RA) Let $f \in C(S^1)$ be a continuous function with a continuous first derivative $f'(x)$. Let $\{a_n\}$ be the Fourier coefficients of f. Prove that $\sum_n |a_n| < \infty$.

3. (DG) Let $S \subset \mathbb{R}^3$ be the surface given as a graph

$$z = ax^2 + 2bxy + cy^2$$

where a, b and c are constants.

(a) Give a formula for the curvature at $(x, y, z) = (0, 0, 0)$ of the induced Riemannian metric on S.

(b) Give a formula for the second fundamental form at $(x, y, z) = (0, 0, 0)$.

(c) Give necessary and sufficient conditions on the constants a, b and c that any curve in S whose image under projection to the (x, y)-plane is a straight line through $(0, 0)$ is a geodesic on S.

4. (AG) Let V and W be complex vector spaces of dimensions m and n respectively, and $A \subset V$ a subspace of dimension l. Let $\mathbb{P}\mathrm{Hom}(V, W) \cong \mathbb{P}^{mn-1}$ be the projective space of nonzero linear maps $\phi : V \to W$ mod scalars, and for any integer $k \leq l$ let

$$\Psi_k = \{\phi : V \to W : \mathrm{rank}(\phi|_A) \leq k\} \subset \mathbb{P}^{mn-1}.$$

Show that Ψ_k is an irreducible subvariety of $\mathbb{P}^{mn-1}$, and find its dimension.

5. (CA) Find a conformal map from the region

$$\Omega = \{z : |z - 1| > 1 \text{ and } |z - 2| < 2\} \subset \mathbb{C}$$

between the two circles $|z - 1| = 1$ and $|z - 2| = 2$ onto the upper-half plane.

6. (A) Let G be a finite group with an automorphism $\sigma : G \to G$. If $\sigma^2 = id$ and the only element fixed by σ is the identity of G, show that G is abelian.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday September 2, 2010 (Day 3)

1. (DG) Let $D \subset \mathbb{R}^2$ be the closed unit disk, with boundary $\partial D \cong S^1$. For any smooth map $\gamma : D \to \mathbb{R}^2$, let $A(\gamma)$ denote the integral over D of the pull-back $\gamma^*(dx \wedge dy)$ of the area 2-form $dx \wedge dy$ on $\mathbb{R}^2$.

(a) Prove that $A(\gamma) = A(\gamma')$ if $\gamma = \gamma'$ on the boundary of D.

(b) Let $\alpha : \partial D \to \mathbb{R}^2$ denote a smooth map, and let $\gamma : D \to \mathbb{R}^2$ denote a smooth map such that $\gamma|_{\partial D} = \alpha$. Give an expression for $A(\gamma)$ as an integral over ∂D of a function that is expressed only in terms of α and its derivatives to various orders.

(c) Give an example of a map γ such that $\gamma^*(dx \wedge dy)$ is a positive multiple of $dx \wedge dy$ at some points and a negative multiple at others.

2. (T) Compute the fundamental group of the space X obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

3. (CA) Let u be a positive harmonic function on $\mathbb{C}$. Show that u is constant.

4. (A) Let $R = \mathbb{Z}[\sqrt{-5}]$. Express the ideal $(6) = 6R \subset R$ as a product of prime ideals in R.

5. (AG) Let $Q \subset \mathbb{P}^5$ be a smooth quadric hypersurface, and $L \subset Q$ a line. Show that there are exactly two 2-planes $\Lambda \cong \mathbb{P}^2 \subset \mathbb{P}^5$ contained in Q and containing L.

6. (RA) Let $\mathcal{C}^\infty$ denote the space of smooth, real valued functions on the closed interval $I = [0, 1]$. Let $\mathbb{H}$ denote the completion of $\mathcal{C}^\infty$ using the norm whose square is the functional

$$f \mapsto \int_I \left((\frac{df}{dt})^2 + f^2 \right) dt.$$

(a) Prove that the map of $\mathcal{C}^\infty$ to itself given by $f \mapsto T(f)$ with

$$T(f)(t) = \int_0^t f(s) ds$$

extends to give a bounded map from $\mathbb{H}$ to $\mathbb{H}$, and prove that the norm of T is 1.

(b) Prove that T is a compact mapping from $\mathbb{H}$ to $\mathbb{H}$

(c) Let $\mathcal{C}^{1/2}$ be the Banach space obtained by completing $\mathcal{C}^\infty$ using the norm given by

$$f \mapsto \sup_{t \neq t'} \frac{|f(t) - f(t')|}{|t - t'|^{1/2}} + \sup_t |f(t)|.$$

Prove that the inclusion of $\mathcal{C}^\infty$ into $\mathbb{H}$ and into $\mathcal{C}^{1/2}$ extends to give a bounded, linear map from $\mathbb{H}$ to $\mathcal{C}^{1/2}$.

(d) Give an example of a sequence in $\mathbb{H}$ such that all elements have norm 1 and such that there are no convergent subsequences in $\mathcal{C}^{1/2}$.