QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2009 (Day 1)

1. (RA) Let H be a Hilbert space and $\{u_i\}$ an orthonormal basis for H. Assume that $\{x_i\}$ is a sequence of vectors such that

$$\sum ||x_n - u_n||^2 < 1.$$

Prove that the linear span of $\{x_i\}$ is dense in H.

2. (T) Let $\mathbb{CP}^n$ be complex projective n-space.

(a) Describe the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$ and, using the Kunneth formula, the cohomology ring $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$.

(b) Let $\Delta \subset \mathbb{CP}^n \times \mathbb{CP}^n$ be the diagonal, and $\delta = i_*[\Delta] \in H_{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ the image of the fundamental class of Δ under the inclusion $i : \Delta \to \mathbb{CP}^n \times \mathbb{CP}^n$. In terms of your description of $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ above, find the Poincaré dual $\delta^* \in H^{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ of δ.

3. (AG) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety, $\mathbb{G}(1, n)$ the Grassmannian of lines in $\mathbb{P}^n$, and $F \subset \mathbb{G}(1, n)$ the variety of lines contained in X.

(a) If X has dimension k, show that

$$\dim F \leq 2k - 2,$$

with equality holding if and only if $X \subset \mathbb{P}^n$ is a k-plane.

(b) Find an example of a projective variety $X \subset \mathbb{P}^n$ with $\dim X = \dim F = 3$.

4. (CA) Let $\Omega \subset \mathbb{C}$ be the open set

$$\Omega = \{z : |z| < 2 \text{ and } |z - 1| > 1\}.$$

Give a conformal isomorphism between Ω and the unit disc $\Delta = \{z : |z| < 1\}$.

5. (A) Suppose ϕ is an endomorphism of a 10-dimensional vector space over $\mathbb{Q}$ with the following properties.

1. The characteristic polynomial is $(x - 2)^4(x^2 - 3)^3$.
2. The minimal polynomial is $(x - 2)^2(x^2 - 3)^2$.
3. The endomorphism $\phi - 2I$, where I is the identity map, is of rank 8.

Find the Jordan canonical form for ϕ.

6. (DG) Let $\gamma : (0, 1) \to \mathbb{R}^3$ be a smooth arc, with $\gamma' \neq 0$ everywhere.

(a) Define the *curvature* and *torsion* of the arc.

(b) Characterize all such arcs for which the curvature and torsion are constant.

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Wednesday September 2, 2009 (Day 2)

1. (CA) Let $\Delta = \{z : |z| < 1\} \subset \mathbb{C}$ be the unit disc, and $\Delta^* = \Delta \setminus \{0\}$ the punctured disc. A holomorphic function f on Δ^* is said to have an *essential singularity* at 0 if $z^n f(z)$ does not extend to a holomorphic function on Δ for any n.

Show that if f has an essential singularity at 0, then f assumes values arbitrarily close to every complex number in any neighborhood of 0—that is, for any $w \in \mathbb{C}$ and $\forall \epsilon$ and $\delta > 0$, there exists $z \in \Delta^*$ with

$$|z| < \delta \quad \text{and} \quad |f(z) - w| < \epsilon.$$

2. (AG) Let $S \subset \mathbb{P}^3$ be a smooth algebraic surface of degree d, and $S^* \subset \mathbb{P}^{3*}$ the *dual surface*, that is, the locus of tangent planes to S.

(a) Show that no plane $H \subset \mathbb{P}^3$ is tangent to S everywhere along a curve, and deduce that S^* is indeed a surface.

(b) Assuming that a general tangent plane to S is tangent at only one point (this is true in characteristic 0), find the degree of S^*.

3. (T) Let $X = S^1 \vee S^1$ be a figure 8, $p \in X$ the point of attachment, and let α and $\beta : [0,1] \to X$ be loops with base point p (that is, such that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = p$) tracing out the two halves of X. Let Y be the CW complex formed by attaching two 2-discs to X, with attaching maps homotopic to

$$\alpha^2\beta \quad \text{and} \quad \alpha\beta^2.$$

(a) Find the homology groups $H_i(Y, \mathbb{Z})$.

(b) Find the homology groups $H_i(Y, \mathbb{Z}/3)$.

4. (DG) Let $f = f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be smooth, and let $S \subset \mathbb{R}^3$ be the graph of f, with the Riemannian metric ds^2 induced by the standard metric on $\mathbb{R}^3$. Denote the volume form on S by ω.

(a) Show that

$$\omega = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}.$$

(b) Find the curvature of the metric ds^2 on S

5. (RA) Suppose that $\mathcal{O} \subset \mathbb{R}^2$ is an open set with finite Lebesgue measure. Prove that the boundary of the closure of $\mathcal{O}$ has Lebesgue measure 0.

6. (A) Let R be the ring of integers in the field $\mathbb{Q}(\sqrt{-5})$, and S the ring of integers in the field $\mathbb{Q}(\sqrt{-19})$.

(a) Show that R is not a principal ideal domain

(b) Show that S is a principal ideal domain

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Thursday September 3, 2009 (Day 3)

1. (A) Let $c \in \mathbb{Z}$ be an integer not divisible by 3.

(a) Show that the polynomial $f(x) = x^3 - x + c \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

(b) Show that the Galois group of f is the symmetric group $\mathfrak{S}_3$.

2. (CA) Let τ_1 and $\tau_2 \in \mathbb{C}$ be a pair of complex numbers, independent over $\mathbb{R}$, and $\Lambda = \mathbb{Z}\langle\tau_1, \tau_2\rangle \subset \mathbb{C}$ the lattice of integral linear combinations of τ_1 and τ_2. An entire meromorphic function f is said to be *doubly periodic* with respect to Λ if

$$f(z + \tau_1) = f(z + \tau_2) = f(z) \quad \forall z \in \mathbb{C}.$$

(a) Show that an entire holomorphic function doubly periodic with respect to Λ is constant.

(b) Suppose now that f is an entire meromorphic function doubly periodic with respect to Λ, and that f is either holomorphic or has one simple pole in the closed parallelogram

$$\{a\tau_1 + b\tau_2 : a, b \in [0, 1] \subset \mathbb{R}\}.$$

Show that f is constant.

3. (DG) Let M and N be smooth manifolds, and let $\pi : M \times N \to N$ be the projection; let α be a differential k-form on $M \times N$. Show that α has the form $\pi^*\omega$ for some k-form ω on N if and only if the contraction $\iota_X(\alpha) = 0$ and the derivative $\mathcal{L}_X(\alpha) = 0$ for any vector field X on $M \times N$ whose value at every point is in the kernel of the differential $d\pi$.

4. (RA) Show that the Banach space ℓ^p can be embedded as a summand in $L^p(0, 1)$—in other words, that $L^p(0, 1)$ is isomorphic as a Banach space to the direct sum of ℓ^p and another Banach space.

5. (T) Find the fundamental groups of the following spaces:

(a) $SL_2(\mathbb{R})$

(b) $SL_2(\mathbb{C})$

(c) $SO_3(\mathbb{C})$

6. (AG) Let $X \subset \mathbb{A}^n$ be an affine algebraic variety of pure dimension r over a field K of characteristic 0.

(a) Show that the locus $X_{\text{sing}} \subset X$ of singular points of X is a closed subvariety.

(b) Show that X_{sing} is a proper subvariety of X.