QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday October 1, 2002 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1a. Exhibit a polynomial of degree three with rational coefficients whose Galois group over the field of rational numbers is cyclic of order three.

2a. The Catenoid C is the surface of revolution in $\mathbb{R}^3$ of the curve $x = \cosh(z)$ about the z axis. The Helicoid H is the surface in $\mathbb{R}^3$ generated by straight lines parallel to the xy plane that meet both the z axis and the helix

$$t \longmapsto [\cos(t), \sin(t), t].$$

(Recall that $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.)

(i) Show that both C and H are manifolds by exhibiting natural coordinates on each.

(ii) In the coordinates above, write the local expressions for the metrics g_C and g_H, induced by $\mathbb{R}^3$, on C and H, respectively.

(iii) Is there a covering map from H to C that is a local isometry?

3a. In $\mathbb{R}^n$, consider the Laplace equation

$$u_{11} + u_{22} + \cdots + u_{nn} = 0.$$

Show that the equation is invariant under orthonormal transformations. Find all rotationally symmetric solutions to this equation. (Here u_{ii} denotes the second derivative in the ith coordinate of a function u.)

4a. Let C denote the unit circle in $\mathbb{C}$. Evaluate

$$\oint_C \frac{e^{1/z}}{1 - 2z}$$

5a. Let $\mathbb{G}(1,3)$ be the Grassmannian variety of lines in $\mathbb{C}P^3$.

(i) Show that the subset $I \subset \mathbb{G}(1,3)^2$

$$I = \{(l_1, l_2) \mid l_1 \cap l_2 \neq \emptyset\}$$

is irreducible in the Zariski topology. (Hint: Consider the space of triples $(l_1, l_2, p) \in \mathbb{G}(1,3)^2 \times \mathbb{C}P^3$ such that $p \in l_1 \cap l_2$, and consider two appropriate projections.)

(ii) Show that the subset $J \subset \mathbb{G}(1,3)^3$

$$J = \{(l_1, l_2, l_3) \mid l_1 \cap l_2 \neq \emptyset, \;\; l_2 \cap l_3 \neq \emptyset, \;\; l_3 \cap l_1 \neq \emptyset\}$$

is reducible. How many irreducible components does it have?

6a. For the purposes of this problem, a manifold is a CW complex which is locally homeomorphic to $\mathbb{R}^n$. (In particular, it has no boundary.)

(i) Show that a connected simply-connected compact 2-manifold is homotopy equivalent to S^2. (Do not use the classification of surfaces.)

(ii) Let M be a connected simply-connected compact orientable 3-manifold. Compute $\pi_3(M)$.

(iii) Show that a connected simply-connected compact orientable 3-manifold is homotopy equivalent to S^3.

(iv) Find a simply-connected compact 4-manifold that is not homotopy equivalent to S^4.

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Wednesday October 2, 2002 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1b. Let $\mathbb{C}[S_4]$ be the complex group ring of the symmetric group S_4. For $n \geq 1$ let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ matrices with complex entries. Prove that the algebra $\mathbb{C}[S_4]$ is isomorphic to a direct sum

$$\bigoplus_{i=1,\dots,t} M_{n_i}(\mathbb{C})$$

and calculate the n_i's.

2b. (i) Show that the 2 dimensional sphere S^2 is an analytic manifold by exhibiting an atlas for which the change of coordinate functions are analytic functions. Write the local expression of the standard metric on S^2 in the above coordinates.

(ii) Put a metric on $\mathbb{R}^2$ such that the corresponding curvature is equal to 1. Is this metric complete?

3b. Let $C \in \mathbb{C}P^2$ be a smooth projective curve of degree $d \geq 2$. Let $\mathbb{C}P^{2*}$ be the dual space of lines in $\mathbb{C}P^2$ and $C^* \subset \mathbb{C}P^{2*}$ the dual curve of lines tangent to C. Find the degree of C^*. (Hint: Project from a point.)

4b. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that the set of points $x \in \mathbb{R}$ where f is continuous is a countable intersection of open sets.

5b. Prove that the only meromorphic functions $f(z)$ on $\mathbb{C} \cup \{\infty\}$ are rational functions.

6b. (i) Show that the fundamental group of a Lie group is abelian.

(ii) Find $\pi_1(\mathrm{SL}_2(\mathbb{R}))$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday October 3, 2002 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let

$$H = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

For $e_2 = (0, 1) \in \mathbb{R}^2$, map H to B by the following diffeomorphism.

$$\mathbf{v} \longmapsto \mathbf{x} = -e_2 + \frac{2(\mathbf{v} + e_2)}{\|\mathbf{v} + e_2\|^2}.$$

(i) Verify that the image of the above map is indeed B. (Hint: Think of the standard inversion in the circle.)

(ii) Consider the following metric on B:

$$g = \frac{dx^2 + dy^2}{(1 - \|\mathbf{x}\|^2)^2}.$$

Put a metric on H such that the above map is an isometry.

(iii) Show that H is complete.

2c. Let $C \subset \mathbb{C}P^2$ be a smooth projective curve of degree 4.

(i) Find the genus of C and give the Riemann-Roch formula for the dimension of the space of sections of a line bundle M of degree d on the curve C.

(ii) If $l \in \mathbb{C}P^2$ is a line meeting C at four distinct points $p_1, \ldots, p_4$, prove that there exists a nonzero holomorphic differential form on C vanishing at the four points p_i. (Hint: Note that $\mathcal{O}_{\mathbb{C}P^2}(1)$ restricted to C is a line bundle of degree 4. Use the Riemann-Roch formula to prove that this restriction is the canonical line bundle K_C.)

3c. Let A be the ring of real-valued continuous functions on the unit interval $[0, 1]$. Construct (with proof) an ideal in A which is not finitely generated.

4c. Construct a holomorphic function $f(z)$ on $\mathbb{C}$ satisfying the following two conditions:

(i) For every algebraic number z, the image $f(z)$ is algebraic.

(ii) $f(z)$ is not a polynomial.

(Hint: The algebraic numbers are countable.)

5c. Let $q < p$ be two prime numbers and $N(q, p)$ the number of distinct isomorphy types of groups of order pq. What can you say, more concretely, about the number $N(q, p)$?

6c. Let $i : S^1 \hookrightarrow S^3$ be a smooth embedding of S^1 in S^3. Let X denote the complement of the image of i. Compute the homology groups $H_*(X; \mathbb{Z})$.