QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday October 2, 2001 (Day 1)

Each question is worth 10 points, and parts of questions are of equal weight.

1a. Let $X$ be a measure space with measure $\mu$. Let $f \in L^1(X, \mu)$. Prove that for each $\epsilon > 0$ there exists $\delta > 0$ such that if $A$ is a measurable set with $\mu(A) < \delta$, then

$$\int_A |f|d\mu < \epsilon.$$ 

2a. Let $P$ be a point of an algebraic curve $C$ of genus $g$. Prove that any divisor $D$ with $\text{deg} D = 0$ is equivalent to a divisor of the form $E - gP$, where $E > 0$.

3a. Let $f$ be a function that is analytic on the annulus $1 \leq |z| \leq 2$ and assume that $|f(z)|$ is constant on each circle of the boundary of the annulus. Show that $f$ can be meromorphically continued to $\mathbb{C} - \{0\}$.

4a. Prove that the rings $\mathbb{C}[x,y]/(x^2 - y^m)$, $m = 1, 2, 3, 4$, are all non-isomorphic.

5a. Show that the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ is not isometric to any sphere $x^2 + y^2 + z^2 = r$.

6a. For each of the properties $P_1$ through $P_4$ listed below either show the existence of a CW complex $X$ with those properties or else show that there doesn’t exist such a CW complex.

P1. The fundamental group of $X$ is isomorphic to $\text{SL}(2, \mathbb{Z})$.

P2. The cohomology ring $H^*(X, \mathbb{Z})$ is isomorphic to the graded ring freely generated by one element in degree 2.

P3. The CW complex $X$ is “finite” (i.e., is built out of a finite number of cells) and the cohomology ring of its universal covering space is not finitely generated.

P4. The cohomology ring $H^*(X, \mathbb{Z})$ is generated by its elements of degree 1 and has nontrivial elements of degree 100.
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1b. Prove that a general surface of degree 4 in \( \mathbb{P}^3_\mathbb{C} \) contains no lines.

2b. Let \( R \) be a ring. We say that Fermat’s last theorem is false in \( R \) if there exists \( x, y, z \in R \) and \( n \in \mathbb{Z} \) with \( n \geq 3 \) such that \( x^n + y^n = z^n \) and \( xyz \neq 0 \). For which prime numbers \( p \) is Fermat’s last theorem false in the residue class ring \( \mathbb{Z}/p\mathbb{Z} \)?

3b. Compute the integral

\[
\int_0^\infty \frac{\cos(x)}{1 + x^2} \, dx.
\]

4b. Let \( R = \mathbb{Z}[x]/(f) \), where \( f = x^4 - x^3 + x^2 - 2x + 4 \). Let \( I = 3R \) be the principal ideal of \( R \) generated by 3. Find all prime ideals \( \mathfrak{p} \) of \( R \) that contain \( I \). (Give generators for each \( \mathfrak{p} \).)

5b. Let \( \mathfrak{S}_4 \) be the symmetric group on four letters. Give the character table of \( \mathfrak{S}_4 \), and explain how you computed it.

6b. Let \( X \subset \mathbb{R}^2 \) and let \( f : X \to \mathbb{R}^2 \) be distance non-increasing. Show that \( f \) extends to a distance non-increasing map \( \hat{f} : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \hat{f}|_X = f \). Does your construction of \( \hat{f} \) necessarily use the Axiom of Choice?

(Hint: Imagine that \( X \) consists of 3 points. How would you extend \( f \) to \( X \cup \{p\} \) for any 4th point \( p \)?)
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Thursday October 4, 2001 (Day 3)

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1c. Let $S \subset \mathbb{P}^3_\mathbb{C}$ be the surface defined by the equation $XY - ZW = 0$. Find two skew lines on $S$. Prove that $S$ is nonsingular, birationally equivalent to $\mathbb{P}^2_\mathbb{C}$, but not isomorphic to $\mathbb{P}^2_\mathbb{C}$.

2c. Let $f \in \mathbb{C}[z]$ be a degree $n$ polynomial and for any positive real number $R$, let $M(R) = \max_{|z|=R} |f(z)|$. Show that if $R_2 > R_1 > 0$, then

$$\frac{M(R_2)}{R_2^n} \leq \frac{M(R_1)}{R_1^n},$$

with equality being possible only if $f(z) = Cz^n$, for some constant $C$.

3c. Describe, as a direct sum of cyclic groups, the cokernel of $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by left multiplication by the matrix

$$
\begin{bmatrix}
3 & 5 & 21 \\
3 & 10 & 14 \\
-24 & -65 & -126
\end{bmatrix}.
$$

4c. Let $X$ and $Y$ be compact orientable 2-manifolds of genus $g$ and $h$, respectively, and let $f : X \to Y$ be any continuous map. Assuming that the degree of $f$ is nonzero (that is, the induced map $f^* : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is nonzero), show that $g \geq h$.

5c. Use the Rouche’s theorem to show that the equation $ze^{\lambda-z} = 1$, where $\lambda$ is a given real number greater than 1, has exactly one root in the disk $|z| < 1$. Show that this root is real.

6c. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function such that for all $x$ and $y \neq 0$,

$$\frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|} \leq B,$$

for some finite constant $B$. Prove that for all $x \neq y$,

$$|f(x) - f(y)| \leq M \cdot |x-y| \cdot \left(1 + \log^+ \left(\frac{1}{|x-y|}\right)\right),$$

where $M$ depends on $B$ and $\|f\|_\infty$, and $\log^+(x) = \max(0, \log x)$. 

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