On the Langlands correspondence for symplectic motives

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In this paper, we present a refinement of the global Langlands correspondence for discrete symplectic motives of rank $2n$ over $\mathbb{Q}$. To such a motive Langlands conjecturally associates a generic, automorphic representation $\pi$ of the split orthogonal group $SO_{2n+1}$ over $\mathbb{Q}$, which appears with multiplicity one in the cuspidal spectrum. Using the local theory of generic representations of odd orthogonal groups, we define a new vector $F$ in this representation, which is the tensor product of local test vectors for the Whittaker functionals [9]. I hope that the defining properties of $F$ will make it easier to investigate the Langlands correspondence computationally, especially for the cohomology of algebraic curves.

Our refinement is similar to the refinement that Weil [24] proposed for the conjecture that elliptic curves over $\mathbb{Q}$ are modular. Namely, Weil proposed that such a curve should be associated with a homomorphic newform $F = \sum a_n q^n$ of weight 2 on $\Gamma_0(N)$, where $N$ is equal to the conductor of the curve.

This paper expands on a letter that I wrote to Serre in 2010. It was motivated by a question Serre posed at my 60th birthday conference, and a suggestion Brumer made of a family of discrete subgroups generalizing $\Gamma_0(N)$. I would like to thank them, and to thank Deligne for his comments.

1 Symplectic motives

Let $M$ be a pure motive of weight $-1$ and rank $2n$ over $\mathbb{Q}$ with a non-degenerate symplectic polarization

$$\psi : \wedge^2 M \rightarrow \mathbb{Q}(1).$$

Some examples come from the primitive odd cohomology groups of geometrically irreducible, nonsingular, projective varieties $X$ over $\mathbb{Q}$. Let $r = \dim(X)$ and for integers $m$ with $1 \leq (2m - 1) \leq r$ consider the motive

$$M = H^{2m-1}(X)_{\text{prim}}(m).$$

The symplectic polarization $\psi$ of $M$ depends on the choice of an ample line bundle $L$ on $X$, or more precisely on the first Chern class $c = c(L)$ of this line bundle in $H^2(X)(1)$. We have the formula

$$\psi(x, y) = \text{Tr}(x \cup y \cup c^{r-(2m-1)}),$$

where $\text{Tr}$ is the isomorphism $\text{Tr} : H^{2r}(r + 1) \rightarrow \mathbb{Q}(1)$. 
The simplest examples come from complete, non-singular curves $X$ of genus $n$ over $\mathbb{Q}$, where $M = H^1(X)(1) = H_1(X)$. Other examples come from non-singular hypersurfaces $X$ in $\mathbb{P}^{2m}$ over $\mathbb{Q}$. In this case we have $M = H^{2m-1}(X)(m)$, as all of the cohomology in the middle (odd) dimension is primitive. In both cases, $H^2(X)(1)$ has rank 1 and the associated polarizations of $M$ are unique up to scaling. In general, for $m = 1$ we have $M = H^1(X)(1) = H_1(A)$, where $A$ is the Albanese variety of $X$. In this case, one needs to choose a polarization of $A$.

We say the symplectic motive $(M, \psi)$ is discrete if its automorphism group $C = \text{Aut}(M, \psi)$ is a finite group scheme over $\mathbb{Q}$. The $\ell$-adic realization $M_\ell$ of $M$ gives a continuous Galois representation $\rho_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{CSp}_{2n}(\mathbb{Q}_\ell)$ with similitude character the $\ell$-adic cyclotomic character. The discrete condition should be equivalent to the assumption that the centralizer in the symplectic group of the image is finite. When $C$ is a finite group scheme, the group $C(\overline{\mathbb{Q}})$ is an elementary abelian 2-group, of rank $\leq n$.

We will see that the discrete condition is automatic if all of the Hodge numbers $h^{p,q} = \dim M^{p,q}$ satisfy $h^{p,q} \leq 1$. However when $M = H_1(A)$ for an abelian variety $A$ of dimension $n \geq 2$, the only non-zero Hodge numbers are $h^{0,-1} = h^{-1,0} = n$ and the discrete condition puts constraints on the endomorphism ring of $A$ over $\mathbb{Q}$. We need to assume this condition to insure that the automorphic representation $\pi$ of $\text{SO}_{2n+1}$ which Langlands conjecturally associates to $M$ is in the discrete spectrum. This implies that $\pi$ is cuspidal, as the local component $\pi_\infty$ is tempered [23]. (Since $M$ is pure of weight $-1$, the local components $\pi_p$ at finite primes should also be tempered, but this is still conjectural at the primes of bad reduction.)

The conjectured functional equation of the complete $L$-function $\Lambda(M,s) = L_\infty(M,s)L(M,s)$ takes the form (cf. [5])

$$\Lambda(M,s) = \epsilon(M)N^{-s}\Lambda(M,-s)$$

where $N \geq 1$ is the Artin conductor of the $\ell$-adic representation $M_\ell$ and $\epsilon(M) = \pm 1$ is the global epsilon factor (both of which are conjecturally independent of $\ell$). We note that the local epsilon factors $\epsilon_v(M)$ are also equal to $\pm 1$, once the additive characters and Haar measure have been normalized properly [8]. The central point $s = 0$ is always critical for $L(M,s)$, in the sense of Deligne [5].

2 Hodge numbers

The Betti realization has a Hodge decomposition

$$M_B \otimes \mathbb{C} = \sum M^{p,q}$$

with $p + q = -1$ and $\dim M^{p,q} = \dim M^{q,p}$. This determines a symplectic representation of the Weil group $W_{\mathbb{R}}$, which is the sum of irreducible 2-dimensional representations induced from characters of the subgroup $W_{\mathbb{C}} \cong \mathbb{C}^*$

$$M_B \otimes \mathbb{C} = \sum_{i=1}^n \text{Ind}(z/\overline{z})^{\alpha_i}.$$
Here the $\alpha_i$ are non-negative half-integers which are determined by the formula

$$2\alpha_i = |p_i - q_i|$$

We order the $\alpha_i$ so that

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n.$$  

If the half-integers $\alpha_i$ determined by the Hodge numbers of $M$ are all distinct, then the centralizer of the image of the Weil group of $\mathbb{R}$ in $\text{Sp}(M_B \otimes \mathbb{C})$ is the finite group scheme $(\mu_2)^n$. Since the automorphism group of $(M, \psi)$ is contained in this centralizer, such a symplectic motive is necessarily discrete. At the other extreme, if $(A, \psi)$ be a polarized abelian variety of dimension $n$ over $\mathbb{Q}$ and $(H_1(A), \psi)$ is the associated polarized symplectic motive $M$ of rank $2n$, then $\dim M^{-1,0} = \dim M^{0,-1} = n$ and the $n$ non-negative half-integers $\alpha_i$ associated to $M$ are all equal to $1/2$.

Let $\lambda_i = \alpha_i + \frac{1}{2}$, so

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq 1.$$  

Then the local $L$-factor and $\epsilon$-factor of $M$ at the real place are given by the formulae

$$L_\infty(M, s) = \prod_\Gamma C(s + \lambda_i)$$

$$\epsilon_\infty(M) = (-1)^\sum \lambda_i$$

3 Generic representations

We let $G = \text{SO}_{2n+1}$ be the split semi-simple group over $\mathbb{Z}$ of adjoint type $B_n$. This is the special orthogonal group of the lattice $\Lambda$ with $\mathbb{Z}$-basis $\{a_1, a_2, \ldots, a_n; c; b_n, b_{n-1}, \ldots, b_1\}$ and symmetric bilinear form $[a_i, b_i] = 1, [c, c] = 2$, and all other inner products zero (so the Gram matrix is anti-diagonal). The integral quadratic form on $\Lambda$ which defines $G$ is given by $q(x) = \frac{1}{2}|x, x|$.

In the following sections, we will associate to the symplectic motive $M$ of rank $2n$ over $\mathbb{Q}$ and a place $v$ of $\mathbb{Q}$ an irreducible, generic representation $\pi_v = \pi_v(M)$ of the local group $G(\mathbb{Q}_v) = \text{SO}_{2n+1}(\mathbb{Q}_v)$.

We first recall what is meant by a generic representation. Let $B$ be the Borel subgroup of $G$ which stabilizes the isotropic flag

$$0 \subset \langle a_1 \rangle \subset \langle a_1, a_2 \rangle \ldots \subset \langle a_1, a_2, \ldots, a_n \rangle$$

Then $B$ stabilizes the complete flag

$$0 \subset \langle a_1 \rangle \subset \ldots \subset \langle a_1, a_2, \ldots, a_n \rangle \subset \langle a_1, a_2, \ldots, a_n, c \rangle \subset \langle a_1, a_2, \ldots, a_n, c, b_n \rangle \subset \ldots \subset \langle a_1, a_2, \ldots, a_n, c, b_n \ldots, b_1 \rangle$$

Let $T$ be the (split) torus in $B$ which stabilizes the opposite isotropic flag

$$0 \subset \langle b_1 \rangle \subset \langle b_1, b_2 \rangle \subset \ldots \subset \langle b_1, b_2, \ldots, b_1 \rangle.$$
Then \( T \) fixes \( c \) and preserves the lines spanned by the \( a_i \) and \( b_i \). Moreover if \( t(a_i) = \tau_i a_i \) then \( t(b_i) = \tau_i^{-1} b_i \). Let \( U \) be the unipotent radical of \( B \) and let \( \eta \) denote the algebraic homomorphism \( U \rightarrow \mathbb{C} \), defined by the sum of inner products

\[
\eta(u) = \sum_{i=1}^{n-1} [u(a_{i+1}), b_i] + [u(c), b_n]/2.
\]

Note that the inner product of \( c \), and hence \( u(c) \), with any lattice vector is even. Since \( G \) is an adjoint group, the stabilizer in \( B \) of the homomorphism \( \eta \) is equal to \( U \).

For \( v \) corresponding to a finite prime \( p \), let \( \chi_v = \chi_p : \mathbb{Q}_p^+ \rightarrow S^1 \) be the non-trivial additive character which is trivial on \( \mathbb{Z}_p \) and given by the formula \( \chi_p(x) = e^{2\pi ix} \). Here \( \mathbb{Q}_p/\mathbb{Z}_p \) is viewed as the subgroup of \( p \)-power torsion in \( \mathbb{Q}/\mathbb{Z} \). For \( v = \infty \) let \( \chi_v = \chi_\infty : \mathbb{R}^+ \rightarrow S^1 \) be the non-trivial additive character which is given by the formula \( \chi_\infty(x) = e^{-2\pi ix} \). Finally, let

\[
\theta_v = \chi_v \circ \eta : U(\mathbb{Q}_v) \rightarrow S^1.
\]

The homomorphism \( \theta_v \) is a continuous, generic character of the locally compact group \( U(\mathbb{Q}_v) \). We say that an irreducible representation \( \pi_v \) of the group \( G(\mathbb{Q}_v) \) is generic when the complex vector space of continuous homomorphisms

\[
\text{Hom}_{U(\mathbb{Q}_v)}(\pi_v, \theta_v)
\]

has dimension one. (When the representation is not generic, the dimension of this space is zero.)

## 4 Real groups

When \( v = \infty \), the representation

\[
M_B \otimes \mathbb{C} = \sum_{i=1}^{n} \text{Ind}(z/\mathbb{Z})^{\alpha_i}
\]

of the Weil group \( W_\mathbb{R} \) corresponds to the Harish-Chandra parameter

\[
\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 1/2
\]

of a generic (limit) discrete series for the group \( G(\mathbb{R}) = \text{SO}(n+1, n) \). Indeed, this parameter is in the closure of the Weyl chamber where all of the simple roots

\[
\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}
\]

are non-compact, and this chamber is the unique one (up to conjugation by the compact Weyl group) corresponding to generic (limit) discrete series [21, Thm 6.2]. We let \( \pi_\infty = \pi_\infty(M) \) be the Casselman-Wallach model [4] of the corresponding irreducible representation of \( G(\mathbb{R}) \). This representation is tempered and lies in the discrete series when all of the \( \alpha_i \) are distinct, and in the limit discrete series.
otherwise. If $\delta$ is the non-trivial character of the component group $G(\mathbb{R})/G^0(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, then the twisted representation $\pi_\infty \otimes \delta$ is isomorphic to $\pi_\infty$.

Let $K$ be the maximal compact subgroup of $G(\mathbb{R})$ which stabilizes the orthogonal decomposition $\Lambda \otimes \mathbb{R} = V^+ \oplus V^-$. Here $V^+$ is the positive definite subspace spanned by the vectors $(a_i + b_i)$ and $c$, and $V^-$ is the negative definite subspace spanned by the vectors $(a_i - b_i)$. Then $K = S(O(V^+) \times O(V^-))$. In the Langlands decomposition $B(\mathbb{R}) = T(\mathbb{R}) U(\mathbb{R}) = MAN$ of the Borel subgroup, the finite subgroup $M = T(\mathbb{R}) \cap K = \langle \pm 1 \rangle^n$ is given by the integral points $T(\mathbb{Z})$ of the split torus and acts by sign changes on the vectors $a_i$ and $b_j$, with the same sign on $a_i$ and $b_i$, and fixes $c$.

In the generic Weyl chamber, we have
\[\rho_n - \rho_c = (1/2, 1/2, \ldots, 1/2).\]

Hence by Schmid’s formula [18, Thm 1.3], the minimal $K$-type $W_\lambda$ of $\pi_\infty$ has highest weight
\[\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 1)\]
with $\lambda_i = \alpha_i + 1/2$. Since $\lambda_1 \geq 1$, the representation $W_\lambda \cong W_\lambda \otimes \delta$ is irreducibly induced from the representation $W_\nu \otimes W_\mu$ of the subgroup $SO(n + 1) \times SO(n)$ of index 2, with
\[\nu = (\lambda_1 \geq \lambda_2 \geq \ldots), \quad \mu = (\lambda_2 \geq \lambda_4 \geq \ldots)\]
This $K$-type $W_\lambda$ occurs with multiplicity one in the irreducible representation $\pi_\infty$ of $G(\mathbb{R})$.

We now define a subgroup $H$ of $K$, which contains $M$ and fixes a unique line in the representation $W_\lambda$. We have an anti-isometry $V^- \to V^+$ taking $(a_i - b_i)$ to $(a_i + b_i)$ whose image is the subgroup stabilizing $c$. This anti-isometry induces an embedding of orthogonal groups $O(V^-) \to O(V^+)$, and hence an embedding
\[O(V^-) \to O(V^+) \times O(V^-)\]
which is the identity on the second factor. The image $H \cong O(n)$ of this diagonal embedding is contained in $K = S(O(V^+) \times O(V^-))$. The finite elementary abelian subgroup $M$ occurring in the Langlands decomposition of the Borel parabolic is contained in $H$, as it induces simultaneous sign changes on the $a_i$ and $b_i$. In fact, $M$ is a Jordan subgroup of $H$.

Since the parameters $\nu$ and $\mu$ of the representations in $W_\lambda = \text{Ind}(W_\nu \otimes W_\mu)$ are interlaced, the classical branching formula implies that the diagonally embedded subgroup $SO(n)$ fixes a unique line in the representation $W_\nu \otimes W_\mu$. Since $W_\lambda$ is irreducibly induced, it follows that the trivial character and the character $\delta$ of $H$ both occur with multiplicity one in $W_\lambda$. We summarize our results in the following proposition.

**Proposition 1**

- There is a generic (limit) discrete series $\pi_\infty$ of $G(\mathbb{R})$ associated to the motive $M$ with Harish-Chandra parameter $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \alpha_n \geq 1/2)$.

- If $\lambda_i = \alpha_i + 1/2$ then the irreducible representation $W_\lambda = \text{Ind}(W_\nu \otimes W_\mu)$ of $K \cong S(O(n + 1) \times O(n))$ occurs with multiplicity one in $\pi_\infty$.

- The representation $W_\lambda$ of $K$ has a unique line fixed by the subgroup $H \cong O(n)$ and a unique line on which $H$ acts by the quadratic character $\delta : O(n) \to \langle \pm 1 \rangle$. 


When \(n = 1\), the real group \(G(\mathbb{R}) = \text{SO}(2, 1)\) is isomorphic to \(\text{PGL}_2(\mathbb{R})\) and the representation \(\pi_\infty\) is a discrete series representation of weight \(2\lambda\). The minimal \(K = \text{O}(2)\) type \(W_\lambda\) has dimension 2, and decomposes into two lines under the action of the simple reflection in \(H = \text{O}(1)\).

In the general case, we define a new vector in \(\pi_\infty\) to be a non-zero vector on the line in the minimal \(K\)-type \(W_\lambda\) on which the subgroup \(H\) acts by the quadratic character \((\delta)^{\sum \lambda_i}\). We recall that the local \(\epsilon\)-factor of \(\pi_\infty\) is given by the formula:

\[
\epsilon_\infty(\pi) = (-1)^{\sum \lambda_i}.
\]

We expect that a generic linear functional for \(U\) and \(\theta_\infty\) is non-zero on a new vector for \(\pi_\infty\). This is true for \(n = 1\) [16].

Although it is not needed in this paper, if one wishes to consider symplectic motives over arbitrary number fields, one needs an analog of a new vector at complex places. In this case, \(\pi_\infty\) is a generic representation of the complex special orthogonal group \(\text{SO}_{2n+1}(\mathbb{C})\) and its minimal \(K\)-type is a representation \(W\) of the compact orthogonal group \(\text{SO}(2n + 1)\). A new vector is a non-zero vector on the unique line in \(W\) fixed by the subgroup \(H = U(n)\) of \(K\).

### 5 \(p\)-adic groups

At a finite prime \(p\) of \(\mathbb{Q}\) and a prime \(\ell \neq p\), the restriction of \(\rho_\ell\) to a decomposition group at \(p\) gives a continuous local Galois representation

\[
\rho_\ell : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{CSp}_{2n}(\mathbb{Q}_\ell).
\]

This gives a complex representation of the Weil-Deligne group of \(\mathbb{Q}_p\)

\[
\phi_{\ell, p} : \text{WD}(\mathbb{Q}_p) \to \text{CSp}_{2n}(\mathbb{C})
\]

which we will assume is independent of \(\ell\).

The similitude character of \(\phi_{\ell, p}\) is the (unramified) cyclotomic character, taking an arithmetic Frobenius element to \(p\). Choosing the positive square root of \(p\) in \(\mathbb{R}\), we may twist this representation by an unramified character to obtain a local Langlands parameter

\[
\phi_p : \text{WD}(\mathbb{Q}_p) \to \text{Sp}_{2n}(\mathbb{C})
\]

for the group \(G = \text{SO}_{2n+1}\). We will assume that this parameter is tempered; this is known to be true when \(p\) does not divide the conductor \(N\) of \(M\), so the local Galois representation is unramified.

Let \(\pi_p\) be the irreducible, tempered, generic representation of \(G(\mathbb{Q}_p)\) which Jiang and Soudry attach to the local parameter \(\phi_p\) [11]. (In fact, the parameter \(\phi_p\) determines an \(L\)-packet of representations, indexed by characters of the component group of the centralizer, and the generic representation \(\pi_p\) corresponds to the trivial character [7].) For almost all primes \(p\), the representation \(\pi_p\) is unramified, so has a unique line fixed by the hyperspecial maximal compact subgroup \(G(\mathbb{Z}_p)\) which stabilizes the lattice \(\Lambda \otimes \mathbb{Z}_p\). We will now describe a compact open subgroup of \(G(\mathbb{Q}_p)\) which fixes a unique line in \(\pi_p\) the general case.

Let \(N \geq 1\) be the global conductor of the motive \(M\), and let \(m = \text{ord}_p(N)\) be the exponent of the prime \(p\) dividing \(N\). Let \(\Lambda(N)\) be the sublattice of \(\Lambda\) which is spanned by the vectors

\[
\{a_1, a_2, \ldots, a_n; Nc; Nb_n, Nb_{n-1}, \ldots, Nb_1\}.
\]
over \( \mathbb{Z} \). The bilinear form \( \langle , \rangle / N \) on \( \Lambda \otimes \mathbb{Q} \) takes integral values on \( \Lambda(N) \). Let \( J_0(p^m) \) be the compact open subgroup of \( G(\mathbb{Q}_p) \) which stabilizes the lattice \( \Lambda(N) \otimes \mathbb{Z}_p = \Lambda(p^m) \otimes \mathbb{Z}_p \). When \( m = 0 \), this is the hyperspecial maximal compact subgroup \( G(\mathbb{Z}_p) \) which stabilizes \( \Lambda \otimes \mathbb{Z}_p \). Following Bruhat and Tits, this can be viewed as the \( \mathbb{Z}_p \) points of a smooth group scheme over \( \mathbb{Z}_p \), which has reduction \( SO_{2n+1}(\mathbb{Z}/p\mathbb{Z}) \) modulo \( p \). When \( m \geq 1 \) the group \( J_0(p^m) \) can also be viewed as a smooth group scheme over \( \mathbb{Z}_p \), and the quotient of its reduction modulo \( p \) by the unipotent radical of the reduction is \( O_{2n}(\mathbb{Z}/p\mathbb{Z}) \), the split orthogonal group of the space spanned by the \( a_i \) and \( b_i \) modulo \( p \). We let \( K(p^m) \) be the subgroup of index 2 with connected reduction \( SO_{2n}(\mathbb{Z}/p\mathbb{Z}) \). Note that \( K(p^m) \) contains the finite group \( T(\mathbb{Z}) \) for all \( p \) and all \( m \).

When \( n = 1 \), \( G = SO_3 \cong \text{PGL}_2 \) and the compact group \( K(p^m) \) is conjugate to the subgroup \( \Gamma_0(p^m) \). When \( n = 2 \), \( G = SO_5 \cong \text{PCSp}_4 \) and the compact subgroups \( K(p^m) \) are conjugate to the paramodular subgroups defined by Roberts and Schmidt [17]. The general definition of the subgroups \( K(p^m) \) was suggested by A. Brumer, based on his extensive computations [2]. We have the following result on generic representations which was proved by Casselman [3] when \( n = 1 \), by Roberts and Schmidt [17] when \( n = 2 \), and by Tsai [20] in general.

**Proposition 2** Let \( \pi_p \) be a generic representation of \( SO_{2n+1}(\mathbb{Q}_p) \) of conductor \( p^m \). Then

- The compact open subgroup \( K_0(p^m) \) fixes a unique line in the representation \( \pi_p \).

- The pairing of 1-dimensional vector spaces \( \text{Hom}_{K_0(p^m)}(\mathbb{C}, \pi_p) \times \text{Hom}_{U(\mathbb{Q}_p)}(\pi_p, \theta_p) \to \mathbb{C} \) is non-degenerate: a non-zero linear form in the space \( \text{Hom}_{U(\mathbb{Q}_p)}(\pi_p, \theta_p) \) is non-zero when restricted to the line fixed by \( K_0(p^m) \) in \( \pi_p \).

- For \( m \geq 1 \) the action of the quotient group \( J_0(p^m)/K_0(p^m) \) on the line of \( K_0(p^m) \) invariants in \( \pi_p \) gives the local \( \epsilon \)-factor \( \epsilon_p(\pi) = \pm 1 \).

We define a new vector in \( \pi_p \) as a non-zero vector on the line fixed by \( K_0(p^m) \). (A similar definition, using the lattice with basis \( \{a_1, \ldots, a_n, \pi^m c, \pi^m b_1, \ldots, \pi^m b_1\} \) works for generic representations of \( SO_{2n+1} \) over a general non-Archimedean local field with uniformizing parameter \( \pi \)).

To give some idea of the conductors of familiar representations, the Steinberg representation of \( SO_{2n+1} \) and its unramified quadratic twist both have conductor \( p^{2n-1} \). This is the smallest conductor for a representation in the discrete series. Depth zero supercuspidal representations all have conductor \( p^{2n} \), which is the smallest conductor for a supercuspidal representation of \( SO_{2n+1} \). In both of these cases, there is an explicit construction of the line fixed by \( K_0(p^m) \) in \( \pi_p \) [20]. The simple supercuspidal representations have minimal positive depth \( 1/(2n) \) and conductor \( p^{2n+1} \), and the epipelagic representations of depth \( 1/2 \) have conductor \( p^{3n} \).

The local generic representation corresponding to a Tate abelian variety of dimension \( n \) with totally multiplicative reduction has conductor \( p^n \), and is only in the discrete series when \( n = 1 \). For \( n > 1 \) this representation is unipotent, but the nilpotent class in its Langlands parameter is a Richardson element in the Siegel parabolic subgroup, and not a regular nilpotent element.
6 Automorphic representations

Let $A$ be the ring of adeles of $\mathbb{Q}$. Since the local representations $\pi_v = \pi_v(M)$ we have attached to the motive $M$ in the previous section are unramified at almost all places $v$, their restricted tensor product

$$\pi = \pi(M) = \bigotimes'_v \pi_v$$

is an irreducible complex representation of $G(A)$.

Since each local representation $\pi_v$ is generic, and the corresponding linear form is non-zero on the spherical vector when $\pi_p$ is unramified, the global representation $\pi$ is generic in the following weak sense. Let $\chi = \prod_v \chi_v : \mathbb{A} \to \mathbb{C}^*$ be the product additive character, which is trivial on the discrete co-compact subgroup $\mathbb{Q} \subset \mathbb{A}$. Define $\theta : U(A)/U(Q) \to \mathbb{C}^*$ by the composition $\theta = \chi \circ \eta$. Then the complex vector space of continuous linear forms

$$\text{Hom}_{U(A)}(\pi, \theta)$$

has dimension 1.

Langlands has conjectured that the representation $\pi$ is also automorphic [12], [13]. Since the symplectic parameter $M$ is discrete and the local representations $\pi_v$ are all tempered, his conjecture is more precise.

**Conjecture 3 (Langlands)** The generic representation $\pi$ associated to $M$ is cuspidal, and appears with multiplicity one in the discrete spectrum of $G$.

Indeed, under the pairing defined in [13], the local character at each place corresponding to the (unique) generic representation in the $L$-packet is trivial. Hence the product of these characters gives the trivial representation of the global centralizer $C = \text{Aut}(M, \psi)$.

In the other direction – starting with an automorphic, cuspidal representation $\pi$ of $SO_{2n+1}$ and producing a symplectic motive $M$ – there has been considerable progress (cf.[1, Thm 2.1.1]) when the infinite component $\pi_\infty$ lies in the discrete series. This generalizes Shimura’s construction of abelian varieties with real multiplication starting from holomorphic cusp forms of weight 2 for $\Gamma_0(N)$ [19].

Assume Langlands’s conjecture, and fix an embedding (which is unique up to scaling) of $\pi$ into the space $\mathcal{A}_0$ of cusp forms on the group $G$. Using this embedding, we conjecture that $\pi$ is generic in the following strong sense. Let $du$ be the Haar measure on $U(A)$ which is counting measure on the discrete subgroup $U(Q)$ and gives the compact quotient $U(A)/U(Q)$ volume 1.

**Conjecture 4** The linear form $\pi \to \mathcal{A}_0 \to \mathbb{C}$ taking a cusp form $f$ in the image of $\pi$ to the integral

$$\int_{U(Q) \backslash U(A)} f(u) \theta(u) du$$

is non-zero, and gives a basis of the one dimensional vector space $\text{Hom}_{U(A)}(\pi, \theta)$.

In fact, we expect this linear form to be non-zero on a specific cusp form $F$ in the image of $\pi$, which we will define in the next section.
7 A global new form

In this section we assume Langlands’s conjecture on the existence of a generic, cuspidal automorphic form \( \pi(M) \) associated to \( M \). We refine it by defining a global new form \( F \) in \( \pi \), made from a tensor product of local new forms.

Let \( z_\infty \) be a new vector on the line where \( H \) acts by the sign character \( \epsilon_\infty = (-1)^{\sum \lambda_i} \) in the minimal \( K \)-type of the local representation \( \pi_\infty \). Let \( z_p \) be a non-zero new vector fixed by \( K_0(p^m) \) in the local representation \( \pi_p \) (on which the normalizer \( J_0(p^m) \) acts by the sign character \( \epsilon_p \)). Finally let \( z = \otimes_v z_v \) be the corresponding vector in the global representation \( \pi \) and let \( F \) be the cusp form which is the image of \( z \) in \( \mathfrak{A}_0 \). We call \( F \) a newform for the representation \( \pi(M) \).

Note that \( F \) is a function
\[
F : G(\mathbb{Q}) \backslash G(\mathbb{A}) / SO(n) \times \prod_p K_0(p^m) \to \mathbb{C}.
\]

Restricting to the infinite component and using strong approximation for the spin group, we see that \( F \) is completely determined by the function
\[
F_\infty : \Gamma_0(N) \backslash G(\mathbb{R}) / SO(n) \to \mathbb{C}
\]
where \( \Gamma_0(N) \) is the arithmetic subgroup of \( G(\mathbb{Q}) \) which is contained in the stabilizer of the lattice \( \Lambda(N) \) and has local components \( K_0(p^m) \). We can also view \( F_\infty \) as a vector valued function
\[
G_\infty : \Gamma_0(N) \backslash G(\mathbb{R}) \to W_\lambda
\]
with \( G_\infty(gk) = k \circ G_\infty(g) \). Taking the inner product with a vector \( z_\infty \) on the appropriate line in \( W_\lambda \) gives back \( F_\infty \).

This only defines the newform \( F \) up to a non-zero complex scalar. We propose to normalize \( F \) by the following condition on its first Fourier coefficient.

**Conjecture 5** Let \( F \) be a newform for the cuspidal representation \( \pi(M) \). Then the integral
\[
a_1(F) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} F(u) \theta(u) du = \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} F_\infty(u) \theta_\infty(u) du
\]
is non-zero.

If this is true, we may normalize the newform \( F \) uniquely by the condition that \( a_1(F) = 1 \).

8 Fourier coefficients

Let \( d = (d_1, d_2, \ldots, d_n) \) be an \( n \)-tuple of integers. Define the homomorphism \( \eta_d : U \to \mathbb{G}_a \) by the formula
\[
\eta_d(u) = \sum_{1 \leq i < n} d_i [u(a_{i+1}), b_i] + d_n [u(c), b_n] / 2.
\]
Let $\theta_d$ be the composition of $\psi$ and $\eta_d$. We define the $d^{th}$ Fourier coefficient of the newform $F$ by the integral

$$a_d(F) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} F(u) \theta_d(u) du = \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} F_\infty(u) \theta_d(u) du.$$ 

The subgroup $M = T(\mathbb{Z})$ is contained in both $\Gamma_0(N)$ and $H$. Therefore, if

$$d^* = (\epsilon_1 d_1, \epsilon_2 d_2, \ldots, \epsilon_n d_n)$$

with each $\epsilon_i = \pm 1$, the corresponding Fourier coefficient

$$a_{d^*}(F) = (\prod \epsilon_i) \sum \lambda_d a_d(F) = \pm a_d(F).$$

If any coordinate $d_i$ of $d$ is equal to zero, we conjecture that $a_d(F) = 0$.

More generally, if $t$ is any element in the rational torus $T(\mathbb{Q})$, we can define the $t^{th}$ Fourier coefficient of the newform $F$ by the integral

$$a_t(F) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} F(u) \theta(t u) du.$$ 

Writing $t = (t_1, t_2, \ldots, t_n)$ as an $n$-tuple of non-zero rational numbers, via the diagonal action of $T$ on the basis elements $(a_1, a_2, \ldots, a_n)$, we obtain our original Fourier coefficients $a_d(F)$ when all of the $t_i = d_i$ are non-zero integers. If any coordinate $t_i$ of $t$ is not an integer, then $a_t(F) = 0$. Indeed, the integral

$$W_t(g) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} F(g) \theta(t u) du$$

defines a function on $G(\mathbb{A})$ on which $U(\mathbb{A})$ acts on the left by the character $\psi_t$ and which satisfies $W_t(1) = a_t(F)$. But the subgroup $B(\mathbb{Z})$ of $B(\mathbb{A})$ is contained in $\prod_p K_0(p^m)$, which fixes $F$ and hence $W_t$ on the right. So for all $g$ in the subgroup $U(\mathbb{Z})$ we have $W_t(g) = W_t(1) = \psi_t(g) W_t(1)$. On the other hand when $t$ is not integral, the character $\psi_t$ is non-trivial on $U(\mathbb{A})$, so $W_t(1) = a_t(F) = 0$.

Since the newform $F$ should be determined by its non-zero integral Fourier coefficients $a_d(F)$, we can attempt to construct it following the ideas of Jacquet and Langlands [10]. Since the representation $\pi = \prod \pi_v$ of $G(\mathbb{A})$ is generic in the weak sense, there is a non-zero continuous linear functional $L : \pi \rightarrow \mathbb{C}$ on which the group $U(\mathbb{A})$ acts by the character $\psi$. Since the new vector is a product of local test vectors for the Whittaker functionals, we may normalize the new vector $v$ in $\pi$ by the condition that $L(v) = 1$. Consider the Whittaker function $W(g) = L(g v)$ on $G(\mathbb{A})$. This is right invariant under the compact subgroup $SO(n) \times \prod_p K_0(p^m)$ and transforms on the left by the character $\psi$ of $U(\mathbb{A})$. Since $\psi$ is trivial on the subgroup $U(\mathbb{Q})$ of $U(\mathbb{A})$, the function $W(g)$ is left invariant by the discrete subgroup $U(\mathbb{Q})$.

The same is true for the functions $W_t(g) = W(t g)$, for all $t$ in the rational torus $T(\mathbb{Q})$. Indeed, $W_t$ transforms on the left by the character $u \rightarrow \psi(t u)$ of $U(\mathbb{A})$. Consider the sum

$$S(g) = \sum_{T(\mathbb{Q})} W_t(g).$$
The function $S(g)$ is right invariant under the compact subgroup $SO(n) \times \prod_p K_0(p^n)$ and by construction left invariant under the discrete subgroup $B(\mathbb{Q}) = T(\mathbb{Q})U(\mathbb{Q})$. Its $t^{th}$ Fourier coefficient, defined by an integral over $U(\mathbb{Q}) \setminus U(\mathbb{A})$ as above, is equal to $W_t(1) = L(t \nu)$, and this coefficient vanishes if any coordinate of $t$ is non-integral. All of this is true for a generic representation $\pi$ of $G(\mathbb{A})$.

To prove that the sum $S$ is equal to the desired newform $F$ and that the global representation $\pi$ is automorphic, one would need to show that the function $S$ on $G(\mathbb{A})$ is left $G(\mathbb{Q})$-invariant. Since it is already left $B(\mathbb{Q})$-invariant by construction, this comes down to checking that $S(wg) = S(g)$ for some element $w$ in $N(T)(\mathbb{Q})$ mapping to the long element $-1$ in the Weyl group of $T$. For example, one can take the transformation $w(a_i) = b_i, w(b_i) = a_i; w(c) = (-1)^n c$. The identity $S(wg) = S(g)$ is presumably equivalent to the functional equation of $\Lambda(M, s)$ (and its twists), as the local $L$-functions are given by an integral transform of the local Whittaker functional on translates of the local new vector $[11]$.

### 9 Abelian varieties

Let $(A, \psi)$ be a polarized abelian variety of dimension $n$ over $\mathbb{Q}$ and let $(H_1(A), \psi)$ be the associated polarized symplectic motive $M$ of rank $2n$. Then $\dim M^{-1,0} = \dim M^{0,-1} = n$ and the $n$ non-negative half-integers $\alpha_i$ associated to $M$ are all equal to $1/2$.

The corresponding real Langlands parameter $W_\mathbb{R} \to \text{Sp}_{2n}(\mathbb{C})$ is given by $n$ copies of the 2-dimensional representation $\text{Ind}_d(\mathbb{Z}/\mathbb{Z})^{1/2}$. The centralizer of the image in $\text{Sp}_{2n}(\mathbb{C})$ is isomorphic to the orthogonal group $O_n(\mathbb{C})$ and has component group of order 2. Hence there are two representations in the corresponding Vogan $L$-packet of limit discrete series. The first is the generic limit discrete series $\pi_\infty$ for $SO(n+1, n)$, which corresponds to the trivial character of the component group. The minimal $K = S(O(n+1) \times O(n))$-type of this representation is the representation

$$W_\lambda = \wedge^m(2m+1) \otimes \wedge^m(2m)$$

when $n = 2m$ is even and the representation

$$W_\lambda = \wedge^{m+1}(2m+2) \otimes \wedge^m(2m+1)$$

when $n = 2m+1$ is odd, as all $\lambda_i = 1$. The local $\epsilon$ factor at infinity is equal to $(-1)^n$.

The second representation in the Vogan $L$-packet is also a limit discrete series, although it is not generic. It gives a representation of the split group $SO(n+1, n)$ when $n = 2m$ is even. When $n = 2$ this is the holomorphic limit discrete series of $SO(3, 2) = \text{PCSp}(4, \mathbb{R})$ (which is the infinite component of holomorphic Siegel modular forms of genus 2 and weight 2). When $n = 2m+1$ is odd the element $-1$ in the center of $\text{Sp}_{2n}(\mathbb{C})$ lies in the non-trivial coset of the connected component of the centralizer and the non-generic (limit) discrete series is a representation of the non-split group $SO(n-1, n+2)$. When $n = 1$ this is the trivial representation of the compact form $SO(3)$.

We need to answer the question as to when the motive $M$ is discrete, as this is not implied by the Hodge numbers. The $\ell$-adic representation $\rho_\ell$ is given by the action on the Tate module of $A$. By
Faltings’ proof of the Tate conjecture [6], the centralizer of the image of $\rho_\ell$ in $\text{Sp}_{2n}(\mathbb{Q}_\ell)$ is given by the $\mathbb{Q}_\ell$ rational points of the group scheme $C = \text{Aut}(A, \psi)$ over $\mathbb{Q}$. Note that

$$C = \{ c \in \text{End}_\mathbb{Q}(A) \otimes \mathbb{Q} : cc^* = 1 \}$$

where $c \rightarrow c^*$ is the Rosati involution of $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ induced by the polarization.

When $A$ is simple over $\mathbb{Q}$, there are four cases to consider [14]:

1. $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ is a totally real field $E$ and $c^* = c$. Then $C = \text{Res}_{E/\mathbb{Q}} \mu_2$.

2. $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ is a totally definite quaternion algebra $D$ over a totally real field $E$ and $c \rightarrow c^*$ is the canonical involution. Then $C = \text{Res}_{E/\mathbb{Q}} H$ where $H$ is the inner form of $\text{SL}_2$ over $E$ associated to $D$.

3. $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ is a totally indefinite quaternion algebra $D$ over a totally real field $E$ and $c \rightarrow c^*$ is the conjugate of the canonical involution by an element $a$ in $D^*$ with $a^2 < 0$ in $F$. Let $K = E(a)$ be the associated CM field. Then $C = \text{Res}_{E/\mathbb{Q}} H$ where $H = O_2(K/E)$.

4. $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ is a division algebra $D$ of rank $n^2$ over the CM field $K$ with totally real subfield $E$ and $c \rightarrow c^*$ is an involution of the second kind on $D$ inducing complex conjugation on $K$. Then $C = \text{Res}_{E/\mathbb{Q}} H$ where $H$ is the inner form of $U_n(K/E)$ associated to the division algebra $D$.

Note that the group scheme $C$ is finite if and only if we are in the first case, when $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ is a totally real field $E$. In the general case, $A$ is isogenous to the product of simple abelian varieties over $\mathbb{Q}$ and we have the following.

**Proposition 6** The symplectic motive $M = H_1(A)$ associated to a polarized abelian variety $A$ over $\mathbb{Q}$ is discrete if and only if the ring $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ is an order in the product $E$ of totally real number fields.

In this case, $C = \text{Aut}(M, \psi) = \text{Res}_{E/\mathbb{Q}} \mu_2$ is a finite commutative group scheme of exponent 2 and order $2^d$, where $d = \text{rank} \text{End}_\mathbb{Q}(A) \leq \text{dim} A$. The group scheme $C$ is equal to the center $\mu_2$ of $\text{Sp}(M)$ if and only if $\text{End}_\mathbb{Q}(A) = \mathbb{Z}$.

Note that when $n = 1$, so $A$ is an elliptic curve, we always have $\text{End}_\mathbb{Q}(A) = \mathbb{Z}$ and the motive $M$ of rank 2 is always discrete with $C = \mu_2$. Our conjecture is essentially equivalent to the refinement of the modularity conjecture, which is due to Weil [24] and was proved by Wiles, Taylor, et alia. It is not identical, as we take the local test vector at infinity on the non-trivial eigenspace for $O(1)$ in the 2-dimensional minimal $K$-type, instead of a “holomorphic” vector on one of the eigenspaces for the subgroup $\text{SO}(2)$ [16].

The computation of the global centralizer $C$ explains the success of Brumer and Kramer [2] in parametrizing abelian surfaces $A$ over $\mathbb{Q}$ with $\text{End}_\mathbb{Q}(A) = \mathbb{Z}$ by holomorphic Siegel modular forms of genus 2 and weight 2, having paramodular level $N$ [15]. The conjectural multiplicity formula of Langlands [12] predicts that such an automorphic representation will appear with multiplicity 1 in the discrete spectrum of $G = \text{SO}_5$ if and only if the product of characters of the component group $C_v$ of the local centralizers has trivial restriction to the global centralizer $C$. These local characters are trivial at all finite primes $p$, as a tempered representation of paramodular level is generic. At the real place, the
character associated to the homomorphic limit discrete series is non-trivial on the component group $C_\infty$ of the local centralizer $O(2)$. But the global centralizer $C = \mu_2$ lies in the connected component of $O(2)$, so the restriction of the product of characters to the global centralizer is trivial.

On the other hand, an abelian surface $A$ with $\text{End}_\mathbb{Q}(A) \otimes \mathbb{Q} = E$, a totally real $\mathbb{Q}$-algebra of rank 2, should correspond to an automorphic representation of $SO_5$ with paramodular level $N$ which is in the generic discrete series at infinity, but not to an automorphic representation with paramodular level $N$ which is in the holomorphic discrete series (as the product of local characters is not trivial on the global centralizer in the latter case). This is compatible with the calculations in [2].

We note that the holomorphic theta series of weight 2 studied by Yoshida [22] have conductor $N = p^2$ but have (non-paramodular) level given by the subgroup of matrices in $\text{P}\text{C}\text{S}p_4(\mathbb{Z})$ with bottom block entry $C$ divisible by $p$. They corresponding to abelian surfaces with endomorphisms by a real quadratic algebra $E$ - the first case is the Jacobian of the modular curve $X_0(23)$. The corresponding local Langlands parameter at $p$ is tamely ramified and corresponds to a sub-regular nilpotent element in $\text{Sp}_4(\mathbb{C})$. Its local centralizer $C_p$ is isomorphic to $O_2(\mathbb{C})$ and the local representation in Yoshida’s holomorphic theta series corresponds to the non-trivial character of the component group at both infinity and $p$. Hence the product of local characters is trivial on the global centralizer $C = \text{Res}_{E/\mathbb{Q}} \mu_2$. In our optic, these abelian surfaces should also give rise to a cusp form on $SO_5$ which is in the generic limit discrete series at infinity and have paramodular level $p^2$.

References


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